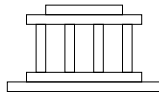


**Fundamentals Of
Heat Transfer**
Theory and Applications

(Class Notes for ME 371)

Michael C. Wendl

*Department of Mechanical Engineering
and School of Medicine
Washington University*



Saint Louis, U.S.A.
(2005)

FUNDAMENTALS OF HEAT TRANSFER THEORY AND APPLICATIONS

©1999, 2003, 2005 by Michael C. Wendl

Permission is granted to copy, distribute and/or display this document under the terms of the Attribution–NoDerivs–NonCommercial License, Version 1.0 or any later version published by Creative Commons. A copy of the license is included in the section entitled “Creative Commons Public License”. For more information, contact Creative Commons, 559 Nathan Abbott Way, Stanford, California 94305, USA.

Non–lawyer summary: this license allows you to freely reproduce, distribute, and display verbatim copies of this document, in electronic or print form, for non–commercial purposes.

This document was implemented entirely with open–source, a.k.a. “free” software: Typeset using L^AT_EX 2 ϵ with graphics by Xfig and plots by XMGRACE, all running on the Linux operating system (Debian distribution)

Document Version 2.1 (August 2005), see Appendix D for revision history

This document contains no color figures

WUSTL–E67–371

Contents

Preface	vi
Chapter 1. Introduction	1
1.1. Physical Mechanisms	2
1.2. Concept of Conservation of Energy	3
1.3. The Continuum Assumption	4
1.4. Absolute Versus Relative Temperature	6
Chapter 2. Elementary Heat Conduction	7
2.1. Fourier's Law	7
2.2. Thermal Properties of Matter	8
2.3. The Conduction Equation	9
2.4. Boundary and Initial Conditions	13
Chapter 3. One-Dimensional Steady Conduction	15
3.1. General Solution	16
3.2. Circuit Analogy	17
3.3. Cylindrical Configurations	21
3.4. Heat Generation	23
3.5. Fin Analysis	27
3.6. Fin Performance Metrics	34
Chapter 4. Transient Conduction	37
4.1. Generalities	37
4.2. Lumped Capacitance Analysis	37
4.3. Applicability of Lumped Capacitance	39
4.4. Casting the General One-Dimensional Problem	42
4.5. Transient Analysis in One Dimension	45
4.6. The Similarity Technique	48
4.7. Multi-dimensional Transient Conduction	50
Chapter 5. Introduction to Convection	53
5.1. Boundary Layer Introduction	54
5.2. Governing Equations	57
5.3. Dimensionless Parameters	60
5.4. Reynolds-Colburn Analogy	63
Chapter 6. External Convection	66

6.1. Laminar Flow Over A Flat Plate	67
6.2. Karman–Pohlhausen Integral Solution	69
6.3. Empirical Correlations	74
Chapter 7. Internal Convection	77
7.1. Laminar Pipe Flow	77
7.2. The Case of Constant Heat Flux	80
7.3. The Couette Problem	85
7.4. Empirical Correlations	88
Chapter 8. Natural Convection	90
8.1. Wall Bounded Convection on a Vertical Flat Surface	91
8.2. Dimensionless Formulation	93
8.3. Similarity Solution for the Vertical Plate	94
8.4. Empirical Correlations	97
Chapter 9. Heat Exchangers	99
9.1. The Thumbnail Diagram	100
9.2. Overall Heat Transfer Coefficient	102
9.3. LMTD Analysis	104
9.4. Correction Factors for Complex Configurations	109
9.5. ϵ -NTU Analysis	110
Chapter 10. Introduction to Radiation	116
10.1. Solid Angle and Radiative Quantities	117
10.2. Blackbody Radiation	122
10.3. Radiation Characteristics of Real Surfaces	126
10.4. Kirchhoff's Law and Gray Surfaces	130
Chapter 11. Radiation Exchange	134
11.1. The View Factor	134
11.2. View Factor Algebra	136
11.3. Blackbody Radiation Exchange	141
11.4. Exchange Among Diffuse Gray Surfaces	141
11.5. The Graybody Matrix Problem	145
11.6. Additional configurations	147
Appendix A. Transient Conduction Example: The Homogeneous Cooling Problem	149
A.1. Separation of Variables Method	149
A.2. Solution Procedure	151
A.3. Determining Mode Coefficients	152
A.4. Example: The Unit Initial Condition	154
Appendix B. Laminar Forced Convection in a Pipe	156
B.1. Volume Flow Rate	156
B.2. Integration of Mean Temperature Equation	156

CONTENTS

v

Appendix C. Blackbody Radiation	158
C.1. Wien's Displacement Law	158
C.2. Radiation Functions	159
Appendix D. Document History	161
About the Author	162
Creative Commons Public License	163
Bibliography	166
Index	170

Preface

Heat transfer is fundamentally important in almost all aspects of engineering, physics, and the life sciences. Issues related to this branch of thermodynamics arise in such varying circumstances as aerospace, biology, consumer products, machinery, medical devices, geo-environmental applications, chemical processes, and nano-technology. There are many excellent texts that introduce this subject and I do not consider this assembly of notes to be any sort of replacement. This merely represents a collection of concepts I consider to be the most valuable in a one-semester introductory course. The material is therefore intended to amplify and augment that which appears in your primary course textbook. In large part, we follow Incropera and Dewitt (2002) in terms of the way material is arranged. A sufficient familiarity with mathematics is presumed, especially calculus and its theorems (e.g. the Chain Rule), linear differential equations, linear algebra, and simple combinatorics. Moreover, much of this material depends upon the understanding of a basic level of fluid mechanics (for example as covered by an introductory course), especially the concepts of conservation of mass and momentum. Problems and examples are minimal, since this volume serves as a collection of notes rather than a complete text. Supplementary margin notes appear at appropriate places suggesting example problems from Incropera and Dewitt (I & D) that could be discussed in class.

The style of presentation leans toward the theoretical and mathematical side of the spectrum rather than the empirical side. My experience is that instilling good mathematical habits early in an undergraduate career is of significant benefit. Moreover, this allows us not to ask readers to take too much on faith. For the most part, we shall prove everything from first principles so that the student can derive a truly fundamental understanding of the material. Where formal proofs are beyond the scope of what might be presented in an introductory format, we will cite appropriate sources. Phenomena which cannot be presented according to first principles, e.g. some aspects of convection, turbulence, etc. will be described in terms of a few empiricisms, but will largely be left for in-class discussion by the instructor. There is no mention of English units of measurement. Where units are necessary, we work exclusively in the International System (SI).

Michael C. Wendl
December 2003

CHAPTER 1

Introduction

Heat Transfer can be described as the process of energy transmission due to a gradient in temperature T , which is a measurable quantity^{1.1}. Thus, determining the temperature distribution for a given problem is the desired solution since other quantities of interest, such as heat flux can be derived from it. The subject of heat transfer, which deals with *non-equilibrium* processes, is essentially an extension of an introductory course in thermodynamics, which is usually limited to *equilibrium* states. For example, a typical problem in thermodynamics might be to determine the final equilibrium T when an annealed steel machine part is quenched in a vat of water. The extension of this problem in heat transfer might be to find the rate of cooling of the steel (T as a function of time and location), which is required to predict the resulting hardness in various regions of the part. What are a few common examples and associated problems and/or analyses?

- Automotive cooling system: implement a design such that power-plant and working fluid temperatures remain in a specified operating range while increasing overall engine efficiency (Taymaz et al., 2003)
- Solar power generation: design and optimize hardware for maximal efficiency in heating a working fluid (Odeh et al., 2000)
- HVAC^{1.2}: size required systems for ultra-high-rise office towers and predict resulting temperature distribution in offices and common spaces (Sinha et al., 1998)
- Biomechanics: analyze body temperature increases as a function of frictional heating by physical exertion (Bergmann et al., 2001)
- Electronics: increase effectiveness of laptop computer cooling systems in response to increasing CPU power dissipation (Pastukhov et al., 2003)

^{1.1}Temperature is a surprisingly difficult concept to define in terms of first principles. This is usually deferred to graduate courses in thermodynamics where it arises from considerations of energy and entropy. Here, we will take the simplistic, but typical approach of understanding temperature merely as a quantity that indicates thermal energy.

^{1.2}Heating, Ventilation, and Air Conditioning

- Medicine: analyze effectiveness of hyperthermia treatments for cancer (Alexander, 2003)
- Aerodynamics: design fuselage to limit effects of aerodynamic heating (Lee et al., 2003)

In summary, it is difficult to find a problem totally devoid of heat transfer considerations.

1.1. Physical Mechanisms

There are basically 3 main modes of heat transfer that we'll study: conduction, convection, and radiation.

- Conduction occurs through microscopic mechanisms, such as lattice vibrations and electron movement. There is no *bulk* motion of the medium — it is strictly a diffusion process. Example: the heat felt when holding one end of a long copper bar in a open fire.
- Convection relies on bulk motion of the medium. Example: cooling effect realized by standing in front of a fan after sprinting the 400m.
- Radiation occurs by way of electromagnetic phenomena as described by Maxwell's equations and does not necessarily depend upon a medium (energy transfer can occur across a vacuum). Example: heating sensation felt by bare skin in the noon sun.

In the “real world”, a problem often relies on all three modes. For example consider the M-16 rifle in automatic fire mode, so that the temperature in the barrel is approximately constant (Fig. 1.1). Heat from expanding

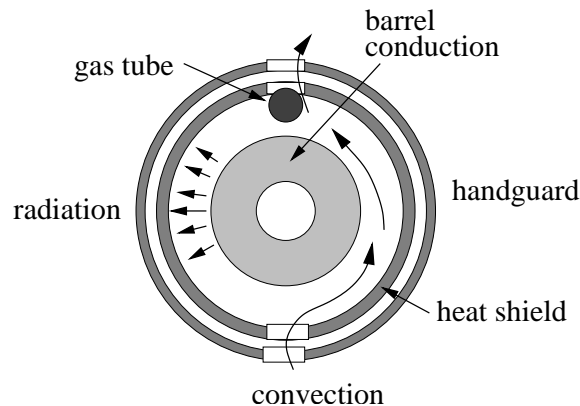


FIGURE 1.1. *M-16 cross section showing barrel, gas tube, heat shield, and outer handguard.*

combustion gas in the bore and friction between the bullet and the bore is conducted through the barrel to its outer surface. The barrel is cooled by

convected air as heat is absorbed, causing a change in density and subsequent air movement. This heat is carried away as the air leaves through holes in the top of the handguard. The barrel is further cooled by radiating energy to inner cooling surface inside the handguard.

Analyses of such real-world problems are usually quite difficult. For example, geometry is complicated and, as we shall see, the full governing equations are non-linear partial differential equations^{1,3}. Thus, we invariably require theoretical, experimental, and computational procedures. However, for this course we'll look primarily at idealized "model" systems that can be solved in closed form.

1.2. Concept of Conservation of Energy

The cornerstone of heat transfer is the law of conservation of energy, which is described in terms of a specific volumetric space called the *control volume (CV)* and a bounding surface called the *control surface (CS)* that encloses the *CV* (Fig. 1.2). In later sections, we will also see that

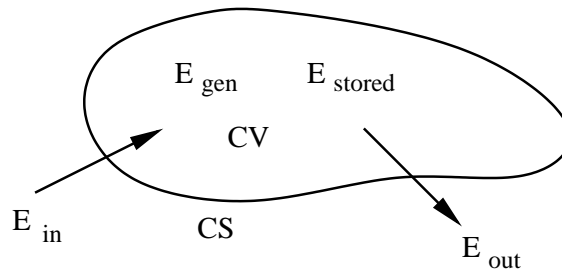


FIGURE 1.2. *Control volume and control surface schematic.*

the conservation laws for mass and momentum arise in certain situations. Conservation of energy can be stated symbolically as

$$(1.1) \quad \frac{dE_{\text{stored}}}{dt} = \dot{E}_{\text{stored}} = \dot{E}_{\text{in}} + \dot{E}_{\text{gen}} - \dot{E}_{\text{out}},$$

where E represents energy and the dot notation connotes a rate process^{1,4}. In other words, the rate of energy increase in the control volume is equal to the rate at which it is generated internally^{1,5} plus the rate at which it comes into the control volume minus the rate at which it leaves. This energy can be of various types — usually we mean thermal energy, but it can also be mechanical work. The terms \dot{E}_{in} and \dot{E}_{out} describe phenomena occurring at (across) the *CS*, while \dot{E}_{gen} is a volumetric term associated with the *CV*.

IBD Ex. 1.3
pp 17

^{1,3}Recall that these same complexities were present in analyzing fluid mechanics problems.

^{1,4}We have not said anything about describing these \dot{E} quantities yet.

^{1,5}Energy can be generated by a variety of means, for example nuclear decay. One of the most common situations is electrical dissipation via current flowing through a wire having non-zero electrical resistance.

Note that the equation integrated over any time period Δt must also hold true:

$$(1.2) \quad \Delta E_{stored} = E_{in} + E_{gen} - E_{out} .$$

There will be occasions where the conservation of energy principle will be required at a surface, for example as a boundary condition at a solid–gas, liquid–gas, or solid–liquid interface (Fig. 1.3). The thickness of this interface

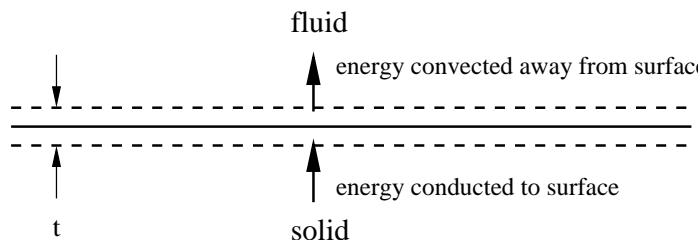


FIGURE 1.3. Conservation of energy at an interface. Dotted lines indicate control surfaces.

t is taken to be vanishingly small, so that the control surface encloses no mass or volume. The storage and generation terms are therefore irrelevant and the conservation of mass equation reduces to

$$(1.3) \quad \dot{E}_{in} = \dot{E}_{out} .$$

Configurations will typically involve conduction on the solid side and convection on the liquid or gas side, so that Eq. (1.3) reduces to the statement that conduction heat transfer equals convection heat transfer at a surface.

1.3. The Continuum Assumption

We will be concerned with heat transfer in both solids and fluids. In fluid mechanics, the so-called *continuum assumption* played an important role in how theory was constructed, especially for gases. For liquids and solids, the continuum assumption is usually taken for granted. Let us have a brief review of this concept.

If we were to look at a mass of fluid at the microscopic level, what we would see are individual molecules interacting with each other. We are not actually interested in the behavior of individual molecules for our engineering applications. Rather, we want to understand the overall (or macroscopic) behavior of the system as a whole. That is, it is the macroscopic properties such as density, temperature, or pressure drop that are of engineering interest. What we are doing from the mathematical perspective is taking averages over small elemental volumes. These volumes must be large enough such that they contain enough molecules at any instant in time to yield a statistically significant average. Yet they must also be small enough so that the statistical average does not vary over the volume — it should be a constant. If these conditions are met, the properties will have definite point

values. In other words, they will be continuous functions of space and time. This is the so-called *continuum assumption*.

the continuum assumption

To illustrate this concept, consider the density ρ of a fluid. A region of fluid is shown in Fig. 1.4(a). We desire to find the density at a point C

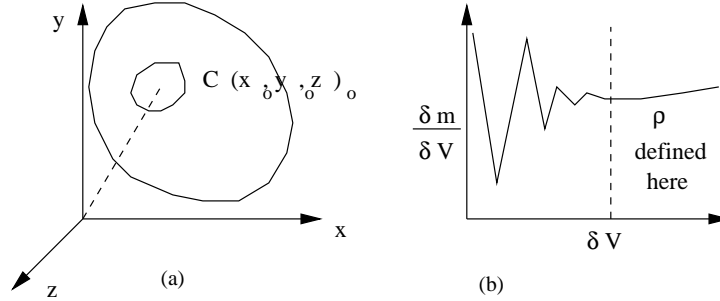


FIGURE 1.4. (a) Defining density as a point function according to the continuum assumption. Large region has volume V and mass m , while the point region C has volume δV and mass δm . (b) Value of density according to size of δV .

located at coordinates (x_0, y_0, z_0) , where density is defined simply as mass per unit volume. For the larger region, the average density within its volume V would simply be $\rho = m/V$. Since this is an average, we would expect ρ to vary within V since V is large. Specifically, the average ρ would not be expected to be equal to the density at point C . To determine the density at C , we compute $\delta m/\delta V$. The question governing the continuum assumption is how small or large is δV ?

Let us plot $\delta m/\delta V$ as a function of δV as in Fig. 1.4(b) starting from large values of δV . The average density tends to approach an asymptotic value as the volume is lowered to the point such as to enclose only homogeneous fluid in the immediate neighborhood of point C . Below this point, the volume is small enough such that the number of molecules contained at a given instant of time is not constant in the average sense. Rather, the number varies erratically, causing a corresponding variation in mass δm , which in turn leads to an erratic value for density $\delta m/\delta V$. There is clearly a lower limit for δV which restricts the continuum assumption. Luckily, this limit is very small compared to the scales we are interested in for engineering calculations^{1.6}. Therefore, we are permitted mathematically to approximate $\delta V \rightarrow 0$ relative to the size of our engineering length scales. By this, we mean that the volume approaches but does not reach 0. Therefore, the continuum assumption will be valid for our problems of interest. Pantou (1984) contains a more detailed discussion.

^{1.6}For example, a cubic meter of air at standard temperature and pressure (15°C, 101.3 kPa) contains about 2.5×10^{25} molecules. Therefore, the number of molecules in a volume about the size of a grain of sand, about 10^{-12} cubic meters, would be 2.5×10^{13} , which is large enough to ensure that the average mass would be constant.

Since the (x_0, y_0, z_0) location of point C is arbitrary, ρ at any point in the region could likewise be determined. If ρ were computed simultaneously for all δV in the fluid, we could formulate an expression as a function of the location and time of the measurements, i.e. $\rho = \rho(x, y, z, t)$. Thus, the continuum assumption leads naturally to functional definitions for the properties of interest.

1.4. Absolute Versus Relative Temperature

We will strictly use the International System (SI), with the most common quantities and units being given in Table 1.1. It is important to remember

TABLE 1.1. Some common SI units

Quantity	Unit
length	meter (m)
mass	kilogram (kg)
time	second (s)
temperature	kelvin (K)

that temperature can be interpreted in an absolute sense, that is, relative to absolute zero, or in a relative sense. The unit of kelvin (K) refers to the absolute scale, while celsius^{1.7} ($^{\circ}\text{C}$) is a relative quantity. Their relationship is given by

$$(1.4) \quad T_{kelvin} = T_{celsius} + 273.15 .$$

Some problems are intrinsically formulated upon the premise of absolute temperature, for example the Ideal Gas Law given by

$$P = \rho RT ,$$

where P is pressure, ρ is fluid density, and R is the ideal gas constant. Similarly, we will see the same requirement for radiation problems, where the absolute temperature raised to the fourth power becomes important. Using relative temperature is incorrect for these problems! Conversely, other problems, for example in convection, depend only on temperature differences, for which it is easy to verify that both relative or absolute units are appropriate^{1.8}.

^{1.7}This is also referred to as centigrade in some countries.

^{1.8}For example, we can express an absolute temperature difference ΔT as

$$\Delta T = T_{k,2} - T_{k,1} = (T_{c,2} + 273.15) - (T_{c,1} + 273.15) = T_{c,2} - T_{c,1} ,$$

which is clearly equivalent to the corresponding relative temperature difference.

CHAPTER 2

Elementary Heat Conduction

Here we will introduce the basic concepts of conduction heat transfer.

2.1. Fourier's Law

In chapter 1, we stated that heat conduction is a diffusion process that occurs via molecular mechanisms. The fundamental governing law is Fourier's Law of Heat Conduction. In one dimension this is^{2.1}

$$(2.1) \quad q'' = -k \frac{dT}{dx},$$

where q'' is heat flux, k is thermal conductivity, and dT/dx is temperature gradient along an independent coordinate x . Thus, temperature is a function of position, and is written in the 1-D case as $T(x)$. Eq. (2.1) is a *phenomenological law* derived from numerous experimental observations rather than first principles. It holds for all the physical configurations we shall study here, but additional terms are required for more complicated materials, for example some animal tissues^{2.2}. Heat flux is clearly a directional quantity. The general three-dimensional form written in vector notation is

$$(2.2) \quad \bar{q}'' = -k \nabla T = -k \left(\hat{i} \frac{\partial T}{\partial x} + \hat{j} \frac{\partial T}{\partial y} + \hat{k} \frac{\partial T}{\partial z} \right).$$

Although the negative sign in Eqs. (2.1) and (2.2) looks rather strange, it is required for consistency of the physics. Heat energy is conducted along a temperature gradient. That is, energy flows from a high temperature region to one of lower temperature. For example, in Fig. 2.1 energy flows in the positive x direction, therefore q'' must be a positive quantity. The temperature gradient is

$$\frac{dT}{dx} = \frac{T_2 - T_1}{x_2 - x_1},$$

which is clearly a negative quantity, since $x_2 - x_1$ is positive, but $T_2 - T_1$ is negative. In fact, $T_2 - T_1$ *must* be negative, otherwise the resulting gradient

^{2.1}We follow the notation of Incropera and Dewitt (2002) where double-prime indicates a flux, whereas lack thereof indicates the quantity has been integrated over an area.

^{2.2}This situation is quite analogous to Newtonian versus Non-Newtonian fluids. Recall that for a Newtonian fluid, shear stress and rate of strain are linearly related, where the constant of proportionality is the viscosity. For Non-Newtonian fluids, additional terms result in much more complicated relationships.

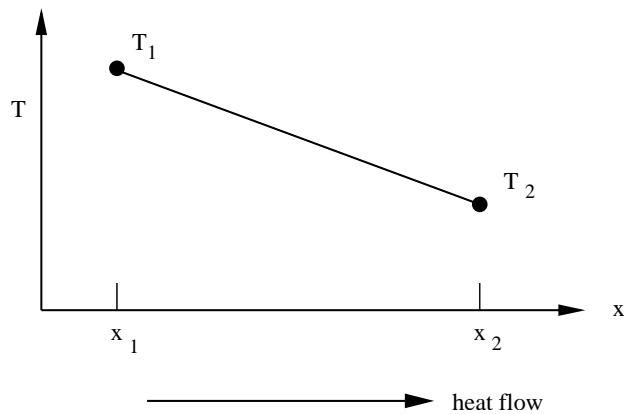


FIGURE 2.1. *Heat is conducted along a negative gradient from high temperature toward low temperature, shown in the positive direction here.*

would not allow energy to flow as shown. Since k is defined as positive, the leading negative sign is clearly required for consistency. In this sense, the temperature gradient is very similar to the pressure gradient in pipe flow: movement proceeds along a negative gradient.

2.2. Thermal Properties of Matter

We mentioned the thermal conductivity k as a material parameter in Fourier's Law. Two other quantities are important to the problems we wish to study: the volumetric heat capacity, ρc_p , and the thermal diffusivity, α . Unites for these quantities are list in Table 2.1. Other properties will arise

TABLE 2.1. Thermal properties

Property	Notation	Units
thermal conductivity	k	$W/(m K)$
volumetric heat capacity	ρc_p	$J/(m^3 K)$
thermal diffusivity	$\alpha = k/(\rho c_p)$	m^2/s

as well, including many of those associated with fluid mechanics problems and surface properties for radiation. These will be introduced as needed.

Recalling that the physical basis for conductivity is at the molecular and atomic levels^{2,3}, we would suspect that solids would generally have the highest conductivities, followed by liquids, and finally gases. This is in fact the case and is largely due to differences in the molecular spacings (Fig. 2.2). As mentioned, the desire for heat transfer problems is to determine the temperature distribution for a given problem. The subject of materials

^{2,3}At the atomic levels, this is primarily associated with the concept of free, or mobile electrons, especially for metals.

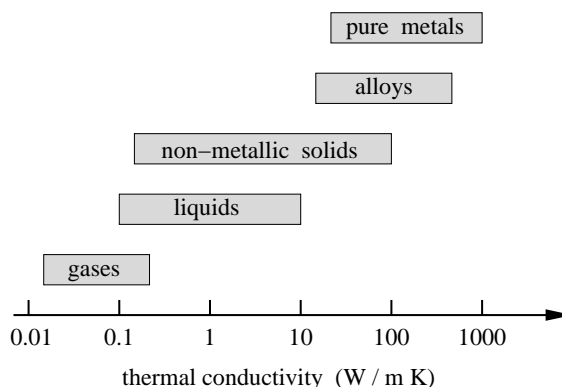


FIGURE 2.2. *Approximate ranges of thermal conductivity for various classifications of matter.*

science deals much more comprehensively with such issues. Actual values are available in many references (e.g. Incropera and Dewitt, 2002, Appendix A).

2.3. The Conduction Equation

It was mentioned in the opening sentences of Chapter 1 that the goal of heat transfer is to determine the temperature distribution. In the general case, this will depend on all three spatial dimensions and on time, so that $T = T(x, y, z, t)$. We introduced Fourier's Law in Eq. (2.1), but this alone does not provide a foundation for calculating T . Instead, we must base our theoretical framework on the conservation law for energy, which was introduced only at a conceptual level in Chapter 1.

We extend the conceptual treatment using the classic differential approach; The resulting equation will be valid for every differential point in a problem domain. This is characteristic of differential formulations: the conservation law must be satisfied simultaneously for all (x, y, z, t) . In this sense, the differential formulation is *exact*, however, it typically presents a more challenging mathematical situation than the integral approach^{2.4}. We define an infinitely small control volume^{2.5} with a properties of interest defined in the center. Any properties depending upon a flux can be extrapolated to the boundaries using truncated 1-term Taylor series^{2.6}.

Define the differential element according to the volume $\delta x \times \delta y \times \delta z$ (Fig. 2.3). According to the concept of conduction, the material is either a solid, or a non-moving fluid. We formulate all quantities at a given instant

^{2.4}The integral approach is not typically as important in heat transfer as in fluid mechanics, but will be introduced later for certain configurations.

^{2.5}Once again, the concept of "infinitely small" remains restricted to the continuum assumption introduced in Chapter 1. A term more suitable is then perhaps "differential", which connotes, small enough, but not too small.

^{2.6}Terms of second-order and higher can all be neglected since they involve products of the differential quantities, e.g. $(\delta x)^2$, which are exceedingly small.

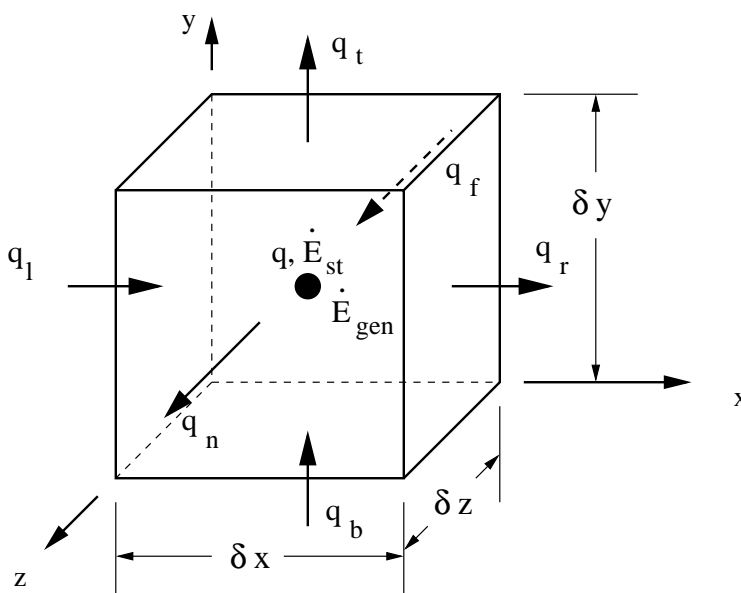


FIGURE 2.3. Differential volume showing energy generation \dot{E}_{gen} and energy storage \dot{E}_{st} terms in the center and heat fluxes at the boundaries.

of time: the energy generation and storage terms, \dot{E}_{gen} and \dot{E}_{st} , represented as volumetric entities in the element, and all heat conduction terms at the boundaries. Expressing these terms as 1-term Taylor series expansions with respect to their associated values defined in the center, we find

$$(2.3) \quad q_l = q_x - \frac{\partial q_x}{\partial x} \frac{\delta x}{2}$$

$$(2.4) \quad q_r = q_x + \frac{\partial q_x}{\partial x} \frac{\delta x}{2}$$

$$(2.5) \quad q_b = q_y - \frac{\partial q_y}{\partial y} \frac{\delta y}{2}$$

$$(2.6) \quad q_t = q_y + \frac{\partial q_y}{\partial y} \frac{\delta y}{2}$$

$$(2.7) \quad q_f = q_z - \frac{\partial q_z}{\partial z} \frac{\delta z}{2}$$

$$(2.8) \quad q_n = q_z + \frac{\partial q_z}{\partial z} \frac{\delta z}{2}$$

Energy generation is simply

$$(2.9) \quad \dot{E}_{gen} = \dot{q} \delta x \delta y \delta z,$$

where \dot{q} is the per unit volume rate of generation^{2.7}. Finally, the rate of change of thermal energy stored in the element can be expressed as

$$(2.10) \quad \dot{E}_{st} = \rho c_p \frac{\partial T}{\partial t} \delta x \delta y \delta z .$$

These terms can be used directly in the conceptual conservation law in Eq. (1.1) on pp. 3, where the rate of energy entering the element is $\dot{E}_{in} = q_b + q_f + q_l$ and the rate of energy leaving is $\dot{E}_{out} = q_n + q_r + q_t$. This operation yields

$$(2.11) \quad \rho c_p \frac{\partial T}{\partial t} \delta x \delta y \delta z = \dot{q} \delta x \delta y \delta z + q_b + q_f + q_l - (q_n + q_r + q_t) ,$$

which becomes

$$(2.12) \quad \rho c_p \frac{\partial T}{\partial t} \delta x \delta y \delta z = \dot{q} \delta x \delta y \delta z + \left(q_y - \frac{\partial q_y}{\partial y} \frac{\delta y}{2} \right) + \left(q_z - \frac{\partial q_z}{\partial z} \frac{\delta z}{2} \right) \\ + \left(q_x - \frac{\partial q_x}{\partial x} \frac{\delta x}{2} \right) - \left(q_z + \frac{\partial q_z}{\partial z} \frac{\delta z}{2} \right) - \left(q_x + \frac{\partial q_x}{\partial x} \frac{\delta x}{2} \right) - \left(q_y + \frac{\partial q_y}{\partial y} \frac{\delta y}{2} \right) ,$$

after substituting Eqs. (2.3) through (2.6) for the heat conduction terms. We then cancel and combine terms appropriately to get

$$(2.13) \quad \rho c_p \frac{\partial T}{\partial t} \delta x \delta y \delta z = \dot{q} \delta x \delta y \delta z - \left(\frac{\partial q_x}{\partial x} \delta x + \frac{\partial q_y}{\partial y} \delta y + \frac{\partial q_z}{\partial z} \delta z \right) .$$

Eq. (2.13) represents the limit in terms of what can be derived strictly from theory. It cannot be solved, because T and q are both unknowns^{2.8}. However, we have additional relationships between T and q in the form of Fourier's Law! In light of Eq. (2.2), we can express the heat terms as

$$(2.14) \quad q_x = -k \frac{\partial T}{\partial x} \delta y \delta z$$

$$(2.15) \quad q_y = -k \frac{\partial T}{\partial y} \delta x \delta z$$

$$(2.16) \quad q_z = -k \frac{\partial T}{\partial z} \delta x \delta y$$

which can be substituted into Eq. (2.13) to obtain

$$(2.17) \quad \rho c_p \frac{\partial T}{\partial t} \delta x \delta y \delta z = \dot{q} \delta x \delta y \delta z - \left[\frac{\partial}{\partial x} \left(-k \frac{\partial T}{\partial x} \delta y \delta z \right) \delta x \right. \\ \left. + \frac{\partial}{\partial y} \left(-k \frac{\partial T}{\partial y} \delta x \delta z \right) \delta y + \frac{\partial}{\partial z} \left(-k \frac{\partial T}{\partial z} \delta x \delta y \right) \delta z \right] .$$

^{2.7}This is the unfortunate notation used in Incropera and Dewitt (2002), which can be easily confused with heat conduction q and heat flux q'' .

^{2.8}We assume that heat generation \dot{q} would be prescribed, or would be measurable for a problem.

All volume terms $\delta x \times \delta y \times \delta z$ cancel and all double-negatives cancel, leaving

$$(2.18) \quad \rho c_p \frac{\partial T}{\partial t} = \dot{q} + \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$$

as the general three-dimensional time-dependent form of the heat conduction equation. We have explicitly derived this equation in the Cartesian (rectangular) coordinate system, which will be sufficient for many of the problems we will study. However, in some instances we will require the cylindrical and spherical forms of the conduction equation. Most reference texts address these cases (e.g. Incropera and Dewitt, 2002).

Eq. (2.18) expresses conservation of energy in terms of the single dependent variable of temperature T . We can solve this in principle because there is one equation and exactly one unknown. In mathematical terms, this is a *linear partial differential equation*, since T does not appear in the form of any products of itself, or its derivatives^{2.9}. Thus, these problems are amenable to theoretical treatments. Non-linear systems (which do exist in many problems) typically require advanced theory and/or computational treatment.

You should have a conceptual understanding of what the terms in this equation mean physically. For example, if we multiply $\partial/\partial x (k \partial T/\partial x)$ in Eq. (2.18) by dx and compare to terms in Eq. (2.11), we see

$$(2.19) \quad \frac{\partial}{\partial x} \left(k \frac{\partial T}{\partial x} \right) dx = q'_l - q'_r$$

is essentially the net heat flux *into* the control volume in the x direction. Similar interpretations hold for the corresponding terms in the y and z directions. Thus, Eq. (2.18) specifies that the net rate of conduction into the control volume plus the rate of generation is equal to the rate of change of energy stored at any point in the domain.

Notice that we have not made any assumptions regarding the conductivity k in Eq. (2.18). However, if k is a constant, it can be moved outside of the derivatives, so that

$$(2.20) \quad \frac{1}{\alpha} \frac{\partial T}{\partial t} = \frac{\dot{q}}{k} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2},$$

where \dot{q} is again the energy generation rate per unit volume (W/m^3) and α is the thermal diffusivity. Mathematically, this is more straightforward to work with than Eq. (2.18). Further simplifications are possible, for example

^{2.9}Actually, the assumption of linearity of the overall equation also depends on the heat generation term \dot{q} as being itself linear. However, if \dot{q} depended upon T in some non-linear fashion, e.g. as a power of T , then the equation would no longer be linear. We shall not discuss such complicated cases here, as the theoretical treatment is quite difficult.

if conduction is also steady (stationary), Eq. (2.20) reduces to

$$(2.21) \quad \frac{\dot{q}}{k} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0.$$

Moreover, if energy generation is absent, we obtain

$$(2.22) \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0,$$

which can also be written $\nabla^2 T = 0$ using the Laplacian operator

$$(2.23) \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Finally, if temperature varies only in one coordinate direction, we obtain

$$(2.24) \quad \frac{d^2 T}{dx^2} = 0,$$

where the regular derivative d has replaced the partial derivative ∂ since temperature is now a function of only a single variable, i.e. $T = T(x)$.

2.4. Boundary and Initial Conditions

The various forms of the conduction equation we have just discussed govern the physics of how heat is conducted in the interior of some pre-defined domain. However, this still does not completely define the problem. We must also have some specification of temperature (or its derivatives) on the boundaries of the domain — these are called *boundary conditions*. Moreover, if the problem is also unsteady, we must also know the *initial condition* for the problem. That is, we must have the value of T (or its derivative) at some specific time.

As you may remember from differential equations, there are two required boundary conditions for each spatial coordinate in the problem since these terms are in the form of second derivatives. Conversely, there is only one initial condition needed for any unsteady problem because the temporal term appears as a first derivative. Almost always, the latter takes the form of prescribing T itself at some instant of time, usually by convention at $t = 0$, which can be written in a few different ways

$$(2.25) \quad T(\vec{x}, t) |_{t=0} = T(\vec{x}, 0) = T_0(\vec{x}),$$

where \vec{x} is one, two, or three spatial coordinates, depending upon the problem, and T_0 is a function of T that gives the temperature distribution throughout the domain at the particular instant of time.

Boundary conditions, on the other hand, can take a number of forms, depending upon the physics of the problem. The most obvious case is similar to the initial condition, where a specific value of T is specified at a coordinate location. For example we could write

$$(2.26) \quad T(\vec{x}, t) |_{x=0} = T_s,$$

where T_s is the temperature at the surface coinciding with the coordinate location $x = 0$. Eq. (2.26) is commonly known as a boundary condition of the first kind, or a *Dirichlet* boundary condition, and is common to heat transfer problems. For example, if the boundary is in contact with a medium undergoing a phase change, the constant temperature boundary condition is a good model.

A boundary condition of the second kind, or *Neumann* boundary condition, involves the derivative of T . For instance, Fourier's Law in Eq. (2.1) on pp. 7 is a ready-made Neumann boundary condition when applied at a physical boundary, e.g.

$$(2.27) \quad -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = q_s''.$$

In other words, the heat flux q_s'' at the $x = 0$ boundary is related to the temperature gradient $\partial T/\partial x$ at $x = 0$ as prescribed by this equation^{2.10}. The special case of the *adiabatic* (perfectly insulated) boundary is given by

$$(2.28) \quad \left. \frac{\partial T}{\partial x} \right|_{x=0} = 0.$$

Boundary conditions of the third type, also called *Robbins* boundary conditions, specify a balance between energy conduction at the boundary and the rate at which this energy is convected away. This case was described conceptually in Fig. 1.3 and Eq. (1.3) on pp. 4 in Chapter 1. The form is something of a combination of the first two types of boundary condition in that it contains both temperature itself and its derivative

$$(2.29) \quad -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = h (T_\infty - T|_{x=0}),$$

where T_∞ is a temperature characterizing the fluid which convects heat away and h is a convection coefficient which we discuss at length in later chapters^{2.11}.

^{2.10}Note that we have replaced the regular derivative d with the partial derivative ∂ to indicate that the temperature may be multi-dimensional, whereas we had not yet introduced this concept in Eq. (2.1).

^{2.11}For the moment, we will assume that h is known for problems involving Eq. (2.29). Later, we will see that convection heat transfer is largely the study of determining h .

CHAPTER 3

One-Dimensional Steady Conduction

Here, we will examine problems where conduction occurs along only one primary dimension, say the x direction, and where the problem is steady. Recall, that Eq. (2.24) on pp. 13 governs this situation, i.e.

$$(3.1) \quad \frac{d^2 T}{dx^2} = 0.$$

This equation is useful for conceptual understanding of conduction, but is also a good engineering approximation for cases where there is dominance in one dimension. For example, consider the two-dimensional domain shown in Fig. 3.1 and assume that the temperature on the boundaries is given by the equations

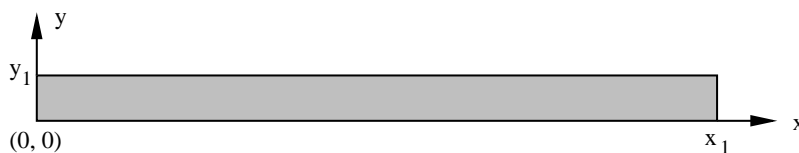


FIGURE 3.1. *Two-dimensional domain with one dimension much longer than the other, i.e. $x_1 \gg y_1$.*

$$T|_{x=0} = T|_{y=0} = 0 \quad \text{and} \quad T|_{x=x_1} = T|_{y=y_1} = T_1.$$

According to discussions in Chapter 2, particularly Eq. (2.22) on pp. 13, we could deduce that this problem is governed formally by the conduction equation

$$(3.2) \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

However, if we look at the relative sizes of temperature gradients using a simple finite-difference approximation,

$$\frac{\partial T}{\partial x} \approx \frac{T_1 - 0}{x_1 - 0} \quad \text{and} \quad \frac{\partial T}{\partial y} \approx \frac{T_1 - 0}{y_1 - 0},$$

it is clear that gradients in the x direction will be small compared to those in the y direction. Thus, such a problem could be satisfactorily approximated as one-dimensional.

3.1. General Solution

The differential equation for the steady one-dimensional heat conduction problem is trivial to solve by direct integration. Integrating Eq. (3.1) once gives

$$(3.3) \quad \frac{dT}{dx} = C_1$$

and integrating once again yields

$$(3.4) \quad T = C_1x + C_2,$$

where C_1 and C_2 are constants to be determined using the boundary conditions. An interesting physical observation can be gleaned for all one-dimensional problems, regardless of their boundary conditions. Specifically, Eq. (3.3) shows that the temperature gradient in the domain is a constant, so that the temperature variation given by Eq. (3.4) is linear. Thus, the temperature distribution is a line whose endpoints are the boundary temperatures and, according to Fourier's Law, the heat flux is constant throughout the domain.

EXAMPLE 3.1:

The heads of the Titanic's 29 coal-fired steam boilers can be modeled as simple circular plates having a radius of $r = 2.4\text{m}$. Assuming a $b = 0.05\text{m}$ plate thickness, a thermal conductivity of $k = 60\text{W/mK}$, a hot-side steam temperature of $T_h = 400\text{K}$, and a stoker room temperature of $T_c = 310\text{K}$, determine the heat transfer through one of the heads.

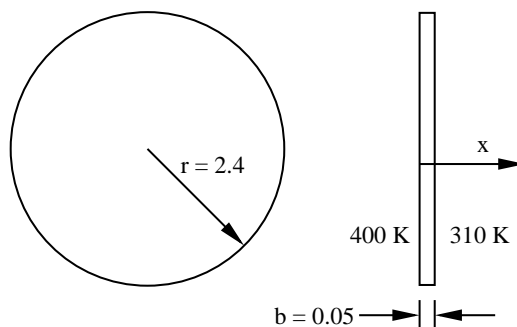


FIGURE 3.2. Frontal and planar view of Titanic steam boiler head.

Gradients in the radial direction will be small compared to gradients through the head since the dimensions differ by about a factor of fifty, therefore, we can treat this problem one-dimensionally. According to the figure, the boundary conditions are $T(0) = T_h$ and $T(b) = T_c$. Using the general solution in Eq. (3.4), we can solve for the constants as $C_2 = T_h$ and

$C_1 = (T_c - T_h)/b$. This gives a temperature distribution of

$$T(x) = \frac{T_c - T_h}{b} x + T_h .$$

The total heat transfer is simply the heat flux times the area of the boiler plate. From Fourier's Law, we obtain

$$q = -k A \frac{dT}{dx} = -k\pi r^2 C_1 = -60 \cdot \pi \cdot 2.4^2 \frac{310 - 400}{0.05} = 1.95 \text{ MW} .$$

Thus, the heat transfer is almost 2 million Watts. $\diamond\diamond\diamond$

3.2. Circuit Analogy

The linear nature of the solution in Eq. (3.4) suggests an analogy between heat conduction and electrical conduction. Recall that in a linear circuit, current I flows along gradients in voltage $\Delta V = V_2 - V_1$ through a resistance R (Fig. 3.3). These entities are related through the simple linear circuit

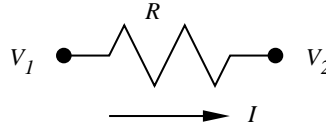


FIGURE 3.3. *Linear circuit showing current flowing through a resistor along a voltage gradient.*

equation

$$(3.5) \quad \Delta V = I R$$

Looking more closely at Eqs. (3.3) and (3.4), we see that the linear drop in temperature gradient coupled with Fourier's Law allows us to write the conduction heat transfer in the exact form as

$$(3.6) \quad q = -k A \frac{dT}{dx} = -k A \frac{T_2 - T_1}{L} = k A \frac{T_1 - T_2}{L} = \Delta T \frac{k A}{L} ,$$

where the generic problem is defined according to boundary conditions $T(0) = T_1$ and $T(L) = T_2$. A little algebra shows the equivalent relationship

$$(3.7) \quad \Delta T = q \frac{L}{k A} = q R_t ,$$

whose form is identical to Eq. (3.5) where ΔT is analogous to voltage (driving potential), q is analogous to current (what "flows" in the circuit), and R_t is a *thermal resistance*. More specifically, R_t in Eq. (3.7) is the thermal resistance of conduction, for which we can use the slightly more descriptive notation

$$(3.8) \quad R_{t,cond} = \frac{L}{k A} .$$

In Eq. (2.29) on pp. 14, we introduced the idea of convection heat transfer as quantified by a convection coefficient h . This equation is a form of *Newton's Law of Cooling*

$$(3.9) \quad q = h A (T|_{x=0} - T_\infty),$$

which shows that this form of convection is also linear. Again, some simple algebra allows us to arrange this equation as

$$(3.10) \quad T|_{x=0} - T_\infty = q \frac{1}{h A}.$$

Or, put more directly

$$(3.11) \quad \Delta T = q R_{t,conv},$$

where

$$(3.12) \quad R_{t,conv} = \frac{1}{h A}$$

is the thermal resistance for convection.

Are such analogs useful for practical problem solving? To answer this question, refer back to the idea at the beginning of this chapter where we discussed the pure one-dimensional problem versus the multi-dimensional problem for which the size of the terms allowed us to use a one-dimensional approximation. The former case is easily analyzed by solving Eq. (3.1) and applying boundary conditions. But how can we treat the latter situation, for example with respect to a problem having composite layers all of different thermal conductivities as shown in Fig. 3.4? Here, the top and bottom

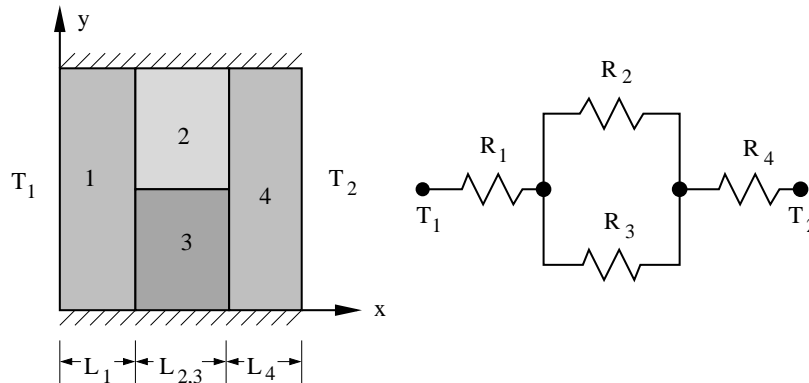


FIGURE 3.4. *One-dimensional problem in a composite domain along with corresponding thermal circuit model.*

boundaries are insulated, and temperature variation is only important in the x direction. So the problem is one-dimensional, but cannot conveniently be treated using Eq. (3.1). However, it readily yields to the circuit analogy. Specifically, we need only reduce the analogous circuit diagram to the simplest possible arrangement using the standard rules of circuit analysis:

- Resistors in series have additive resistance, e.g. for N resistors in series

$$R_{t,total} = R_{t,1} + R_{t,1} + \cdots + R_{t,N}$$

- Resistors in parallel are governed by the reciprocal relationship, e.g. for N resistors in parallel

$$\frac{1}{R_{t,total}} = \frac{1}{R_{t,1}} + \frac{1}{R_{t,1}} + \cdots + \frac{1}{R_{t,N}}$$

These resistances are of the appropriate variety: conductive as given by Eq. (3.8) or convective, as in Eq. (3.12). Nodes function exactly as in circuit analysis as well. For example, the temperature at the interface between two layers can be computed from the corresponding nodal value in the circuit diagram. Thus, the entire conduction problem can be solved using the standard tools from circuit analysis. Moreover, the circuit model allows us to neatly account for other factors, such as the large drop in temperature across the interface of between two materials. Microscopically, an interface contains small gas-filled voids that may present a significant thermal “contact resistance” $R_{t,c}$, which we would expect to be able to determine based on the two materials (Incropera and Dewitt, 2002). Lets look at an example that combines all of these phenomena.

EXAMPLE 3.2:

A manufacturer of pre-fabricated components for high-rise buildings is determining the thermal rating of a new panel design. The system consists of a header made up of an exterior cladding material having thermal conductivity k_c , and an inner insulator of conductivity k_i , window glass with conductivity k_g , and a footer of the same construction as the header. It can be assumed that heat transfer occurs one-dimensionally through the panel. What would the thermal circuit for this system be assuming that there is a thermal contact resistance of R_c between the cladding and insulator components?

The panel height and width are $w \times L$ and the header, pane, and footer each have a height of $w/3$. The total thickness is t , half of which is cladding and half of which is insulating material. The following factors must be accounted for

- convection boundary condition that relates the temperature of the outside air T_∞ to the temperature of the panel’s outer surface $T_{s,o}$ through a known convection coefficient h — based on a total surface area of wL , Eq. (3.12) indicates this resistance would be

$$R_{conv} = \frac{1}{w L h}$$

- heat transfer through the header and footer, which is resisted by the cladding, the insulator, and the contact resistance between them, which we assume is known to be R_c — these three resistances are

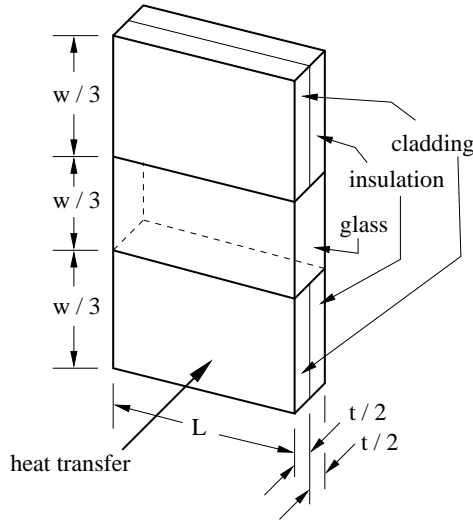


FIGURE 3.5. *Pre-fabricated window panel for a high-rise building.*

in series, so they are additive, with the conductive resistances for cladding and insulators given by Eq. (3.8). The total for each of the header and footer is

$$R_{h,f} = \frac{t/2}{k_c L w/3} \Big|_{cladding} + R_c + \frac{t/2}{k_i L w/3} \Big|_{insulation}$$

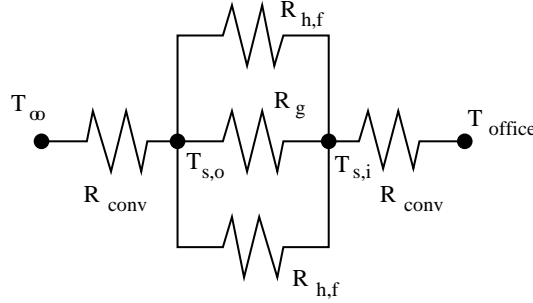
- heat transfer through the glass pane, whose resistor is also given by Eq. (3.8) as

$$R_g = \frac{t}{k_g L w/3} \Big|_{cladding}$$

where we note the the full thickness t appears in the numerator since the glass is twice as thick as the cladding and insulation.

- convection boundary condition that relates the temperature of the inside office environment T_{office} to the temperature of the panel's inner surface $T_{s,i}$ through a known convection coefficient h — since the area and the coefficient are the same as for the external side, this resistor is given by R_{conv} above

We can assemble these into the circuit diagram shown in Fig. 3.6, where the nodes give the appropriate temperatures. This can easily be reduced to a single circuit via the rules for resistors in parallel and in series. $\diamond\diamond\diamond$

FIGURE 3.6. *Circuit model for pre-fabricated window panel.*

3.3. Cylindrical Configurations

As with the Cartesian system, the circuit analogy is applicable to other coordinate systems if a one-dimensional model of conduction is justified. If we were to go through the same derivation of the conduction equation in cylindrical coordinates (r, θ, z) as shown back in Chapter 2, we would find

$$(3.13) \quad \rho c_p \frac{\partial T}{\partial t} = \dot{q} + \frac{1}{r} \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right)$$

(see e.g. Incropera and Dewitt, 2002, for derivation). If we assume one-dimensional steady conductivity in the radial direction, a constant thermal conductivity k , and no heat generation, Eq. (3.13) simplifies to

$$(3.14) \quad \frac{k}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) = 0.$$

When integrated twice, this yields

$$(3.15) \quad T(r) = C_1 \ln r + C_2,$$

where C_1 and C_2 are constants to be determined using the boundary conditions.

Assume we have an annular section where $T = T_h$ at $r = r_h$ (an inner “hot” side) and $T = T_c$ at $r = r_c$ (an outer “cool” side) where $r_c > r_h > 0$. Substituting, we get $T_h = C_1 \ln r_h + C_2$ and $T_c = C_1 \ln r_c + C_2$. Therefore, $T_h - T_c = C_1(\ln r_h - \ln r_c) = C_1 \ln(r_h/r_c)$, which can be solved for the constant as

$$(3.16) \quad C_1 = \frac{T_h - T_c}{\ln(r_h/r_c)}.$$

Backsubstituting, the value of the other constant is found to be

$$(3.17) \quad C_2 = T_c - \frac{T_h - T_c}{\ln(r_h/r_c)} \ln r_c,$$

so that the resulting temperature distribution is

$$(3.18) \quad T(r) = \frac{(T_h - T_c) \ln(r/r_c)}{\ln(r_h/r_c)} + T_c.$$

Application of the Fourier Law yields

$$(3.19) \quad q = -k A \frac{dT}{dr} = -k \times (2\pi r L) \times \left(\frac{T_h - T_c}{\ln(r_h/r_c)} \frac{r_c}{r} \frac{1}{r_c} \right),$$

where L is the length along the z coordinate direction of the domain. With a little algebra, this can be simplified to

$$(3.20) \quad q = \frac{2\pi L k (T_h - T_c)}{\ln(r_c/r_h)},$$

where we have absorbed the negative sign according to $\ln(r_c/r_h) = -\ln(r_h/r_c)$. Recalling from Eq. (3.7) that the thermal resistance $R_t = \Delta T/q$, we see that the thermal circuit representation for conduction in a cylindrical domain is

$$(3.21) \quad R_t = \frac{\ln(r_c/r_h)}{2\pi L k}.$$

Recall, we have assumed $r_c > r_h > 0$ thus far, so we could write Eq. (3.21) in a slightly more generic form as

$$(3.22) \quad R_t = \frac{\ln(r_o/r_i)}{2\pi L k},$$

where subscripts “o” and “i” denote the outer and inner boundaries, respectively.

Let’s go back for a minute to the M-16 cooling problem. The bores of GI barrels are chrome-plated for increased wear and anti-fouling properties, however, this adds another “layer” of thermal resistance (Fig. 3.7). Let r_i

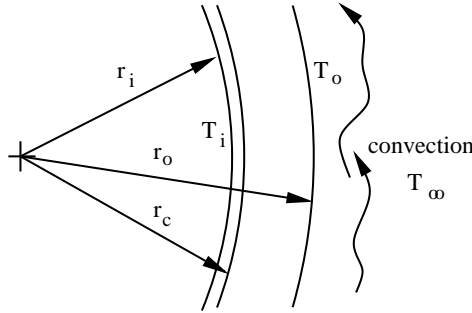


FIGURE 3.7. Chrome-plated GI M-16 barrel.

be the radius of the bore, r_c the radius of the chrome plating, and r_o be the outer radius of the barrel. We assume that the plating is “perfect”, i.e. there are no interstitial voids, so there is no thermal contact resistance. Also let’s assume that conditions in the bore are such that the bore temperature equals the temperature at the inner surface of the barrel. How can this system be represented by a thermal circuit? We simply construct a series circuit representing the thermal resistance of the chrome plating, the barrel itself, and the convection occurring at the outer surface of the barrel, as depicted in Fig. 3.8. Applying the circuit analogy, we find

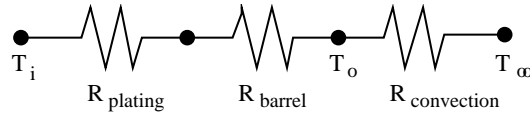


FIGURE 3.8. Circuit for M-16 barrel problem.

$$(3.23) \quad q = \frac{T_i - T_\infty}{R_{plating} + R_{barrel} + R_{convection}},$$

where

$$(3.24) \quad R_{plating} = \frac{\ln(r_c/r_i)}{2\pi L k_{chrome}},$$

$$(3.25) \quad R_{barrel} = \frac{\ln(r_o/r_c)}{2\pi L k_{barrel}},$$

$$(3.26) \quad R_{convection} = \frac{1}{2\pi r_o L h}.$$

Here, L is the length of the barrel and h is the convection coefficient at its outer surface. We notice that for this problem, the thickness of plating is very thin, therefore r_c/r_i is just slightly more than unity. The natural log of this is close to zero, therefore, we conclude that chrome-plating the barrel does not measurably affect the heat transfer situation.

*IED Ex. 3.4
pp 107*

3.4. Heat Generation

We now want to look at the important case of heat generation occurring in the medium itself, e.g. conversion of electrical energy to heat energy through the action of Ohmic resistance. Simplifying Eq. (2.21) for one-dimensional conduction in a Cartesian domain, we obtain

$$(3.27) \quad \frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0,$$

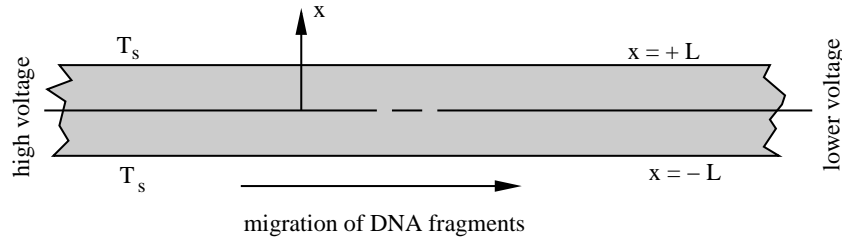
where for example $\dot{q} = I^2 R_{elec}$ per unit volume for the electric problem. This equation is straightforward to solve by direct integration, which yields

$$(3.28) \quad T(x) = -\frac{\dot{q}x^2}{2k} + C_1x + C_2$$

as the general solution. Again, we must use the boundary conditions to evaluate the constants of integration.

Let us consider an application of Eq. (3.28) to the problem of DNA fragment separation using Polyacrylimide Gel Electrophoresis^{3.1}. This technique, used to generate the raw data of the Human Genome Project (Lander et al., 2001), separates fragments by size by exploiting the difference in net charge (Fig. 3.9). The axial voltage difference leads to heat generation within

^{3.1}Fred Sanger won the 1980 Nobel Prize in Chemistry for inventing the sequencing reaction that supplies the input for this process.

FIGURE 3.9. *Polyacrylimide Gel Electrophoresis.*

the gel, which is a significant design consideration in sequencing lab instrumentation. If there is a sufficient temperature rise, DNA will de-nature in the gel. Let the gel thickness be $2L$ and the ohmic dissipation per unit volume be $\dot{q} = I^2 R_{gel}$. Also, let the temperature at both surface of the gel be T_s . We can solve for the constants as $C_1 = 0$ and $C_2 = 0.5 \dot{q} L^2/k + T_s$, so that the exact solution is

$$(3.29) \quad T(x) = \frac{\dot{q}L^2}{2k} \left(1 - \frac{x^2}{L^2} \right) + T_s .$$

This is a parabolic profile with a maximum at $x = 0$ (the center of the gel) of

$$(3.30) \quad T_{max} = \frac{\dot{q}L^2}{2k} + T_s .$$

Eq. (3.30) describes the influences that design parameters have on the process.

- the surface temperature of the gel T_s results from the surrounding temperature in the laboratory environment, which we assume is not a systematically controlled variable
- heat generation \dot{q} results from the voltage gradient and is therefore a function of the experimental requirements — this is maximized to the greatest degree possible to achieve reasonable results
- the gel conductivity k is essentially a fixed parameter
- the gel thickness $2L$ can be directly controlled — gels are therefore made as thin as possible

Since it appears as a square, L is a strong variable, for example halving it reduces the temperature rise by a factor of four.

Solutions are developed in a similar fashion for the cylindrical coordinate system. The governing equation is

$$(3.31) \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) + \frac{\dot{q}}{k} = 0 .$$

Once again, integrating twice, we get the general solution

$$(3.32) \quad T(r) = -\frac{\dot{q} r^2}{4k} + C_1 \ln r + C_2.$$

Now let's look at an interesting combination problem for which the circuit analogy can be applied for part of the problem, but for the other part having heat generation it can't.

EXAMPLE 3.3:

Find the current-carrying capacity of a wire (Fig. 3.10) assuming the failure mode is the onset of insulation melting and find the temperature in the wire at this condition. Assume failure occurs at $T = 330K$. The following parameters are given, where “w” denotes a wire property and “i” denotes an insulation property:

- radii: $r_w = 0.002m$ and $r_i = 0.003m$
- conductivities: $k_w = 80W/(mK)$ and $k_i = 10W/(mK)$
- convection parameters: $T_\infty = 300K$ and $h_i = 20W/(m^2K)$
- electrical resistance: $\rho'_w = 0.008\Omega/m$

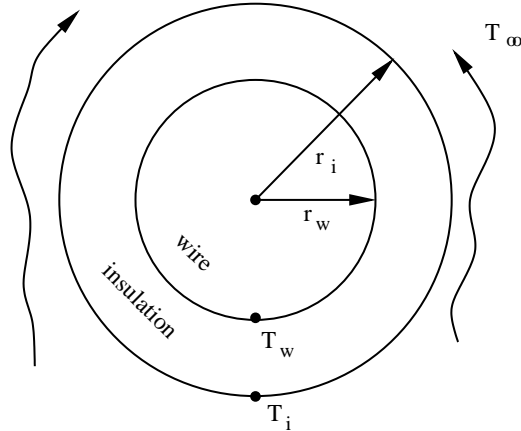


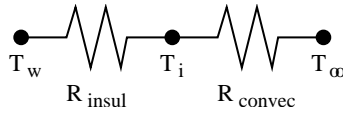
FIGURE 3.10. Electrical power transmission wire.

Current capacity: Intuitively, we know that the maximum insulation temperature will occur at the wire surface r_w because if heat is flowing radially outward, the temperature in the insulation at larger radii *must* be lower. Since there's no energy generation in the insulation, we can use the thermal circuit analogy to solve for the limiting current I . The circuit equation is $q = (T_w - T_\infty)/R_{total}$, where the resistors are added in series as

$$R_{total} = R_{insul} + R_{convec} = \frac{\ln(r_i/r_w)}{2\pi L k_i} + \frac{1}{2\pi r_i L h_i}$$

and L is the length of the wire. This simplifies to

$$R_{total} = \frac{0.00645 + 2.6526}{L} \frac{K}{W} = \frac{2.659}{L} \frac{K}{W}.$$

FIGURE 3.11. *Circuit representing the wire insulation.*

Because the system is steady-state, the heat generated by the dissipation all goes through the insulation, therefore q is simply the dissipation rate, which is $I^2 R_{electric} = 0.008 L I^2 W$. Therefore, from the circuit equation, we have

$$0.008 L I^2 \frac{2.659}{L} = T_w - T_\infty ,$$

where $T_w - T_\infty = 30K$. This results in

$$I^2 = \frac{30}{0.008 \cdot 2.659} ,$$

which yields 37.6 Amps^{3.2}.

Max wire temperature: For the wire, no circuit analogy can be used because of the heat generation. Therefore, we have to refer back to the general solution in Eq. (3.32) for temperature with heat generation:

$$T(r) = -\frac{\dot{q} r^2}{4k} + C_1 \ln r + C_2 .$$

We see that because the wire is a solid cross-section that $C_1 = 0$, otherwise the term would blow up because $\ln 0 \rightarrow -\infty$, so that

$$T(r) = -\frac{\dot{q} r^2}{4k} + C_2 .$$

We have to know the volumetric heat generation rate, \dot{q} , which can be computed as

$$\dot{q} = \frac{\dot{E}_{gen}}{Vol} = \frac{I^2 R_{electric}}{\pi r_w^2 L} = \frac{0.008 L 37.6^2}{\pi 0.002^2 L} = 897,828 \frac{W}{m^3} .$$

Now we can evaluate C_2 using the fact that the temperature at the wire—insulation boundary is known to be 330 K at failure, therefore

$$C_2 = T(0.002) + \frac{\dot{q} r^2}{4k} = 330 + \frac{897,828 \cdot 0.002^2}{4 \cdot 80} = 330.01 K .$$

The exact solution is therefore

$$T(r) = -\frac{\dot{q} r^2}{4k} + 330.01 ,$$

^{3.2}This is about enough to carry the load of forty 100 W light bulbs. Of course, with an appropriate safety factor defined by a municipal electrical code, the maximum prescribed load would be somewhat less.

which allows us to determine the temperature at the center of the wire in a straightforward fashion. Evaluating $T(0)$, we find

$$T(0) = C_2 = 330.01 \text{ K} .$$

We see that there's hardly any difference in temperature between the center of the wire and the edge of the wire. It's essentially at a constant temperature. $\diamond\diamond\diamond$

3.5. Fin Analysis

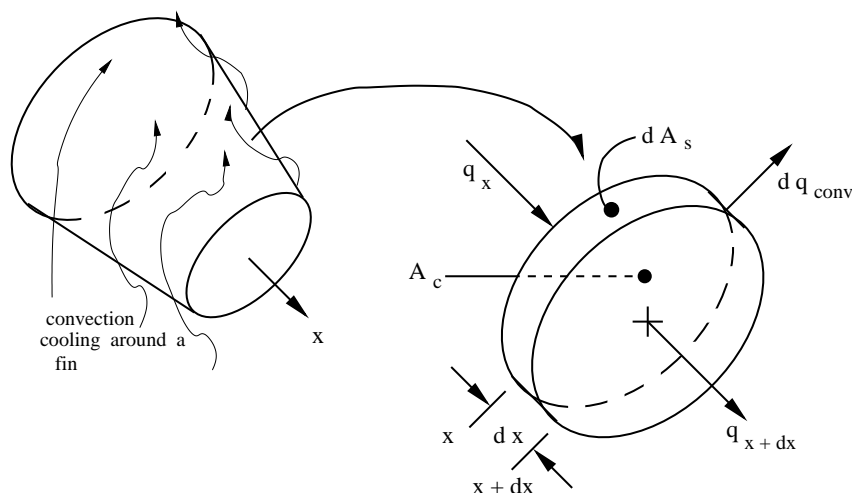
Extended surfaces, or “fins”, are often used to increase heat transfer, for example as applied to air conditioner coils, I/C chips, heat exchangers, car radiators, lawn mower engines, pipes, etc. Why is this the case? If we recall the concept of Newton's Law of Cooling introduced in Eq. (3.9) on pp. 18, we can write a generic form of this convection equation as

$$(3.33) \quad q = h A \left(T|_{surface} - T_{fluid} \right) ,$$

where h is once again the convective heat transfer coefficient. How can the various components of this equation be modified to increase q ?

- T_{fluid} : Usually this cannot be changed much, if at all, for example if air is the working fluid then the atmospheric air temperature is essentially a constant.
- $T|_{surface}$: Similar to T_{fluid} , the device temperature and thus the resulting surface temperature is usually constrained to a small operating range.
- h : This can be increased to a degree, however, it is still highly constrained, for example adding a blower to increase convection may be possible, but it may be impractical due to weight, size restrictions, electrical considerations, aerodynamics, etc.
- A : It's easy to increase this by huge margins using fins!

Of course, to properly analyze such systems, we must first have the appropriate equation describing the conservation of energy. Here, we develop the so-called fin equation, which is a special case of the general conduction equation that explicitly incorporates a convection boundary condition. In the usual fashion, we analyze a differential length dx of a fin as shown in Fig. 3.12. Here, energy is conducted along the fin in the x direction and dissipated by convection along the entire radial boundary. Conduction in this problem is actually two-dimensional, both in the radial direction and along the axis. However, as with the wire problem in the previous example, temperature gradients are small in the radial direction, so that conduction will be correspondingly small enough to be neglected. We therefore assume the

FIGURE 3.12. *Differential fin element.*

conduction is strictly one-dimensional along the axial x direction. We will further assume steady-state conditions, constant conductivity, and constant convection coefficient. The local cross-sectional area at x is A_c , however, we assume this can vary along the axis, so we properly write this as a function of x , as in $A_c(x)$. The local boundary area exposed to convection is dA_s , which is a constant for any particular element.

Heat is conducted into the element at x at a rate of q_x and is conducted out of the element at $x + dx$ at a rate q_{x+dx} . Heat is convected away at the radial boundary at a rate dq_{conv} . For energy to be conserved, we write the equation

$$(3.34) \quad q_x = q_{x+dx} + dq_{conv}.$$

(This is clearly a form of Eq. (1.1) on pp. 3: $\dot{E}_{stored} = \dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out}$, where $\dot{E}_{stored} = \dot{E}_{gen} = 0$.) We immediately recognize q_x from Fourier's Law in Eq. (2.1) on pp. 7, which can be written for this problem as

$$(3.35) \quad q_x = -k A_c(x) \frac{dT}{dx},$$

where $A_c(x)$ is again the cross-sectional area of the fin that varies with x . We can expand q_{x+dx} according to a truncated 1-term Taylor series^{3.3} to obtain

$$(3.36) \quad q_{x+dx} = q_x + \frac{dq_x}{dx} dx = -k A_c(x) \frac{dT}{dx} - k \frac{d}{dx} \left(A_c(x) \frac{dT}{dx} \right) dx,$$

where we have removed k from under the derivative in the second term, since it is assumed constant. Finally, we write the convection term once

^{3.3}Recall we did this with the derivation of the conduction equation in Chapter 2 for flux terms as well.

again according to Newton's Law of Cooling as

$$(3.37) \quad dq_{conv} = h dA_s (T - T_\infty) ,$$

where T_∞ is the temperature of the fluid absorbing the heat. We can now substitute these components back into our original conservation law in Eq. (3.34) to obtain

$$(3.38) \quad h dA_s (T - T_\infty) - k \frac{d}{dx} \left(A_c(x) \frac{dT}{dx} \right) dx = 0 .$$

If we divide by $k dx$ and change sign, we get the canonical form

$$(3.39) \quad \frac{d}{dx} \left(A_c(x) \frac{dT}{dx} \right) - \frac{h}{k} \frac{dA_s}{dx} (T - T_\infty) = 0 .$$

This equation is the *generalized fin equation* for one-dimensional conduction. It is a second-order equation which requires 2 boundary conditions to solve it.

Notice that a non-trivial aspect of Eq. (3.39) is that we must know the function $A_c(x)$ before we can solve the problem. That is, Eq. (3.39) depends on the geometric attribute of how the cross-section varies along the fin. Let us examine the most straightforward case: a uniform cross-section. Under these conditions, the total surface area A_s is simply the perimeter of the cross-section P multiplied by the length x

$$A_s = P x ,$$

so that we obtain $dA_s/dx = P$ by simple differentiation. Moreover, for constant cross-section $A_c(x) \rightarrow A_c$ is a constant and can be moved outside of the differential term in Eq. (3.39). With these two observations, Eq. (3.39) simplifies to

$$(3.40) \quad \frac{d^2 T}{dx^2} - \frac{h P}{k A_c} (T - T_\infty) = 0 .$$

This is a second-order linear non-homogeneous equation, however, it's not as easily integrated as the standard conduction equation because it contains both the second derivative of T and T itself. We recall from differential equations that the typical method of handling this type of equation is (1) solve the homogeneous form then (2) solve to get the particular solution, then (3) sum to get the general solution. However, in this case we are lucky that a change of variables^{3,4} can be made to reduce the problem to a simple second-order homogeneous equation: $\theta(x) = T(x) - T_\infty$. The result is

$$(3.41) \quad \frac{d^2 \theta}{dx^2} - m^2 \theta = 0 ,$$

^{3,4}Since T_∞ is a constant, derivatives of T can be recast directly in terms of θ , for example

$$\frac{dT}{dx} = \frac{d(\theta + T_\infty)}{dx} = \frac{d\theta}{dx} + \frac{dT_\infty}{dx} = \frac{d\theta}{dx} .$$

Further derivatives clearly show the same behavior.

where

$$(3.42) \quad m^2 = \frac{hP}{kA_c}$$

is a composite parameter made up of the physical and geometric attributes of the problem. The general solution for Eq. (3.41) can be written in the form of exponentials as

$$(3.43) \quad \theta(x) = C_1 e^{mx} + C_2 e^{-mx} .$$

Likewise, it could also be written in terms of the hyperbolic trigonometric functions according to the identities

$$(3.44) \quad \cosh \beta = \frac{e^\beta + e^{-\beta}}{2}$$

and

$$(3.45) \quad \sinh \beta = \frac{e^\beta - e^{-\beta}}{2} .$$

Of course, now we are back in the old position of having to use boundary conditions for an actual problem to determine the constants of integration. We once again need 2 boundary conditions, one at the base of the fin $x = 0$, and one at the end of the fin at $x = L$ (Fig. 3.13). The first boundary

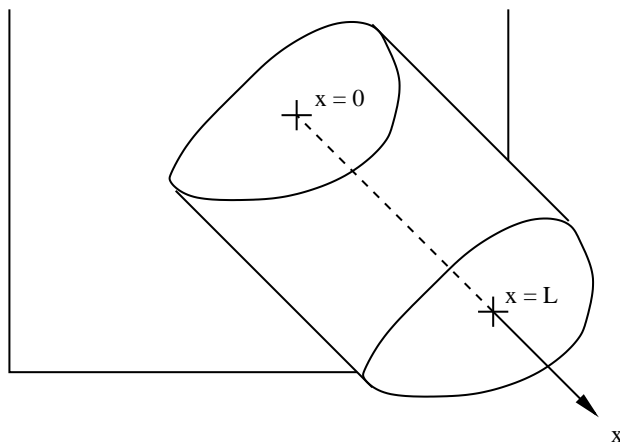


FIGURE 3.13. *Boundary locations for fins.*

condition is rather straightforward: at the base of the fin we will assume that the temperature is known, i.e. $T(0) = T_0$. In our derived variable, this translates to $\theta(0) = T(0) - T_\infty = T_0 - T_\infty = \theta_0$. If we substitute this into Eq. (3.43), we find that one of our boundary equations will always be

$$(3.46) \quad \theta(0) = \theta_0 = C_1 + C_2 .$$

At the end of the fin, there are 4 cases that are of interest here. Listed in order of mathematical difficulty, they are:

- Fin is very long: Fins convect heat to the surrounding fluid along their length. If we assume that the fin tip is very long, i.e. $L \rightarrow \infty$, eventually *all* of the heat will be convected away, so that the temperature at the very end *must* be equal to T_∞ . That is, there is no longer any temperature gradient because there is no more heat to be transferred. According to our original substitution $\theta(x) = T(x) - T_\infty$, this means that $\theta(\infty) \rightarrow 0$. If we substitute this into Eq. (3.43), it is clear that the first term would be unbounded, so we must conclude that $C_1 = 0$. We can then use the boundary condition at the base to find $C_2 = \theta_0$, so that the solution is

$$(3.47) \quad \theta(x) = \theta_0 e^{-mx}.$$

- Negligible fin tip convection: In this case, we assume the end of the fin is insulated, so that $q = 0$ at $x = L$. Writing this in terms of Fourier's Law, we obtain a boundary condition of

$$\left. \frac{d\theta}{dx} \right|_{x=L} = 0.$$

We can take the derivative of Eq. (3.43)

$$\frac{d\theta}{dx} = C_1 m e^{mx} - C_2 m e^{-mx}$$

and substitute this into the boundary condition to obtain the equation

$$C_1 e^{mL} - C_2 e^{-mL} = 0.$$

This equation, along with the boundary condition from $x = 0$ in Eq. (3.46) allows us to solve for the constants. For example, we have $C_1 = \theta_0 - C_2$ from Eq. (3.46), which can be substituted into the above equation

$$\begin{aligned} 0 &= (\theta_0 - C_2) e^{mL} - C_2 e^{-mL} \\ &= \theta_0 e^{mL} - C_2 (e^{mL} + e^{-mL}) \end{aligned}$$

so that C_2 can be solved as

$$C_2 = \theta_0 \frac{e^{mL}}{e^{mL} + e^{-mL}}.$$

We can substitute this right back into Eq. (3.46) to solve for C_1 as

$$C_1 = \theta_0 - \theta_0 \frac{e^{mL}}{e^{mL} + e^{-mL}} = \theta_0 \left(1 - \frac{e^{mL}}{e^{mL} + e^{-mL}} \right)$$

which, with a little algebra, simplifies to

$$C_1 = \theta_0 \frac{e^{-mL}}{e^{mL} + e^{-mL}}.$$

Finally, we can substitute C_1 and C_2 into Eq. (3.43) to obtain the exact solution

$$\begin{aligned}\theta(x) &= \theta_0 \frac{e^{-mL}}{e^{mL} + e^{-mL}} e^{mx} + \theta_0 \frac{e^{mL}}{e^{mL} + e^{-mL}} e^{-mx} \\ &= \theta_0 \frac{e^{-mL} e^{mx} + e^{mL} e^{-mx}}{e^{mL} + e^{-mL}} \\ &= \theta_0 \frac{e^{m(L-x)} + e^{-m(L-x)}}{e^{mL} + e^{-mL}}.\end{aligned}$$

According to the identity for cosh in Eq. (3.44), we can write this result in the more simple form

$$(3.48) \quad \theta(x) = \theta_0 \frac{\cosh m(L-x)}{\cosh mL},$$

which is the exact solution.

- Fin tip temperature is known: That is, we have a standard Dirichlet boundary condition at $x = L$, similar to that at the base $x = 0$

$$\theta(L) = \theta_L.$$

The procedure to evaluate C_1 and C_2 for this case is identical to what we have seen so far, although the algebra is somewhat more involved. The final solution turns out to be

$$(3.49) \quad \theta(x) = \frac{\theta_L \sinh mx + \theta_0 \sinh m(L-x)}{\sinh mL}.$$

- Convection from fin tip: Rate of energy transferred to the fluid by convection at the tip of the fin equals the rate at which energy arrives at the tip via conduction. We write the standard equation

$$h A_c (T(L) - T_\infty) = -k A_c \left. \frac{dT}{dx} \right|_{x=L},$$

which can be simplified using our change of variables to

$$h \theta(L) = -k \left. \frac{d\theta}{dx} \right|_{x=L}.$$

This case is the most tedious to solve. We find

$$(3.50) \quad \theta(x) = \theta_0 \frac{\cosh m(L-x) + \lambda \sinh m(L-x)}{\cosh mL + \lambda \sinh mL},$$

where $\lambda = h/(mk)$ is a constant that depends upon the characteristics of the boundary condition, specifically h and k .

In each of these four cases, we now know the exact temperature distribution for the entire fin in terms of our modified variable $\theta(x)$. In principle, it should now be possible to calculate the total amount of heat transferred by the fin to its surroundings q_f . This is the quantity of primary interest

for many real-world problems. But how do we do this? As is often the case, there is an easy way, and a hard way.

Refer to Fig. 3.14. One of our original assumptions was that the problem

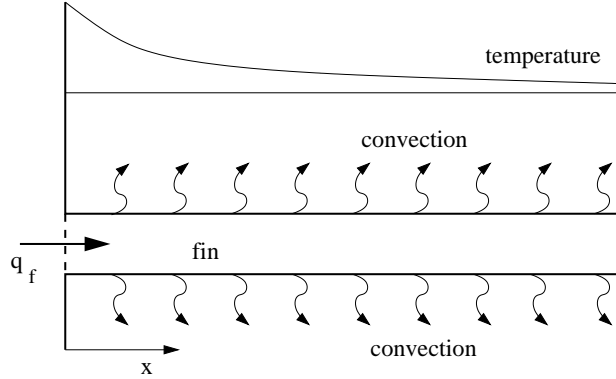


FIGURE 3.14. *Schematic and qualitative temperature distribution in a constant cross-section fin. Magnitude of temperature gradient decreases with increasing x as a consequence of the reduction in q with increasing x due to continuous convection loss from the fin surface.*

is steady state, so that the energy entering the fin via conduction at $x = 0$ must be equal to the energy convected by the fin to its surroundings over its total area, i.e. over $0 \leq x \leq L$. The two possibilities may already be clear at this point.

The hard way is to integrate Newton's Law of Cooling about the entire surface area of the fin

$$q_f = \int_{A_s} h (T(x) - T_\infty) dA_s = \int_{A_s} h \theta(x) dA_s .$$

By definition, this gives us the entire energy transferred to the surroundings. While this task is somewhat straightforward for the infinite fin using Eq. (3.47), the other cases are rather more difficult, especially since the fin tip convection must be considered^{3.5}. Conversely, the easy way is to simply evaluate the total heat transferred into the fin by way of Fourier's Law applied at the base of the fin $x = 0$

$$(3.51) \quad q_f = -k A_c \left. \frac{d\theta}{dx} \right|_{x=0} .$$

Since we know the exact temperature distribution, this is a simple matter of taking a first derivative. For example, in the case of the infinite fin, we find

$$\frac{d\theta}{dx} = -\theta_0 m e^{-mx} ,$$

^{3.5}The exception is of course the case of the insulated fin tip.

which can be evaluated at $x = 0$ and substituted into Eq. (3.51) to yield

$$q_f = -k A_c (-\theta_0 m) = \theta_0 k A_c m .$$

Recalling the original definition of parameter m from Eq.(3.42), we can write the result in the more intuitive form

$$(3.52) \quad q_f = \theta_0 \sqrt{h P k A_c} .$$

This equation indicates that the temperature difference at the base is the strongest parameter of the problem. In particular, doubling the temperature difference will double the heat transfer. Conversely, the other four variables are not as significant, since they appear as square roots. For example, to double the heat transfer, the convection coefficient would have to be increased by a factor of 4, and so forth.

Total heat transfer for the other three cases can be worked out in a similar fashion. If we write the shorthand quantity in Eq. (3.52) as $\xi = \theta_0 \sqrt{h P k A_c}$, we find for the adiabatic (insulated) fin tip

$$(3.53) \quad q_f = \xi \tanh mL ,$$

for the prescribed fin tip temperature boundary condition

$$(3.54) \quad q_f = \xi \frac{\cosh mL - \theta_L/\theta_0}{\sinh mL} ,$$

and for the convective fin tip boundary condition

$$(3.55) \quad q_f = \xi \frac{\sinh mL + \lambda \cosh mL}{\cosh mL + \lambda \sinh mL} ,$$

where again $\lambda = h/(mk)$.

IED Ex. 3.8
pp 133

3.6. Fin Performance Metrics

The fact that heat must be conducted along the fin means that the fin itself represents a thermal resistor in the context of the circuit analogy. In other words, a conductor has been added to an original un-finned surface and this represents another resistance. So how do we know that a fin actually helps increase the heat transfer? We can evaluate this by using a measure of *fin effectiveness*, i.e. the ratio of the finned heat transfer rate, q_f , to the heat transfer that would occur without the fin, $h A_c \theta_0$

$$(3.56) \quad \varepsilon_f = \frac{q_f}{h A_c \theta_0} ,$$

where A_c is cross-sectional area of the virtual fin and θ_0 represents the temperature difference at that location. Generally, the use of fins can be justified if there would be at least a 100 % increase in heat transfer, i.e. $\varepsilon_f \geq 2$ (Incropera and Dewitt, 2002).

If we make what might be a big assumption in some cases, i.e. the coefficient of convection h stays the same whether the fin is present or not, we

can compute ε_f for a variety of cases. For example, we can insert Eq. (3.52) for the infinite fin into Eq. (3.56) to find

$$(3.57) \quad \varepsilon_f = \frac{\theta_0 \sqrt{h P k A_c}}{h A_c \theta_0} = \sqrt{\frac{k P}{h A_c}}.$$

What does this say generally about fins?

- Fin performance is enhanced by using a high conductivity material. Of course, this is almost obvious.
- An effective fin has a high P/A_c ratio, that is the perimeter is large compared to the cross-sectional area. For example, assume the fin has a square cross-section of side length b , then $P = 4b$ and $A_c = b^2$, therefore $P/A_c = 4/b$, which means the ratio, thus the effectiveness improves as b becomes smaller. Therefore, lots of *thin, closely-spaced* fins promote heat transfer, with the provision that the space between them is still enough to maintain good convection.
- The use of fins is justified for conditions when h itself is small, because the increase in heat transfer can be many fold.

Typically h for gases is much lower than h for liquids (e.g. Incropera and Dewitt, 2002, Table 1.1, pp. 8). In particular, unforced convection for a gas is quite low. That's why for cases like the small lawn mower engine, which does not have an active cooling system, there are always fins to augment heat removal. Heat exchanger design also follows such logic. Automotive radiators have lots of fins on the air side (low h), but no fins on the anti-freeze coolant side (high h).

Another metric is *fin efficiency*, i.e. the ratio of the actual heat transfer, q_f , to the maximum possible heat transfer, $h A_f \theta_0$, which would take place if the entire fin was at the base temperature, so that convection was maximized all along the fin length^{3.6}. The form is

$$(3.58) \quad \eta_f = \frac{q_f}{h A_f \theta_0}.$$

Eqs. (3.52) through (3.55) can be used to compute the efficiency for constant-area fins using the various types of boundary conditions we have discussed. Efficiencies for more complicated fins, for example those having variable cross-sectional areas, have been worked out and tabulated in references (e.g. Özişik, 1985; Mills, 1999; Incropera and Dewitt, 2002).

Perhaps more useful is the concept of *overall fin efficiency* for an array of fins. The definition is similar to Eq. (3.58) for a single fin, except quantities are summed for all fins, i.e.

$$(3.59) \quad \eta_0 = \frac{q_{total}}{h A_{total} \theta_0}.$$

^{3.6}Recall from Fig. 3.14 that the gradient along a real fin lessens the heat transfer.

We can derive a relationship between η_f for a single fin and the overall efficiency for N fins as follows. First, the total area A_{total} is simply the sum of the areas of all the fins plus the remaining overall area of the unfinned base A_b . For N fins this is

$$(3.60) \quad A_{total} = NA_f + A_b .$$

According to Eq. (3.58), the heat transfer from one fin is $\eta_f h A_f \theta_0$, so that the heat transfer from N fins is $N \eta_f h A_f \theta_0$. We include the heat transfer from the remaining unfinned base $h A_b \theta_0$, so that the total heat transfer is

$$(3.61) \quad q_{total} = N \eta_f h A_f \theta_0 + h A_b \theta_0 .$$

Now, assuming that h is taken as a single constant value for both the finned surface and the remaining area of the base, we can substitute Eqs. (3.60) and (3.61) into Eq. (3.59) to find

$$(3.62) \quad \eta_0 = \frac{N \eta_f h A_f \theta_0 + h A_b \theta_0}{h (N A_f + A_b) \theta_0} = \frac{N \eta_f A_f + A_b}{N A_f + A_b} .$$

This equation is sometimes given in a slightly different form that can be derived by adding and subtracting 1 from η_f as

$$\eta_0 = \frac{N (\eta_f + 1 - 1) A_f + A_b}{N A_f + A_b} = \frac{[N A_f + A_b] + (\eta_f - 1) N A_f}{N A_f + A_b} ,$$

which simplifies to

$$(3.63) \quad \eta_0 = \frac{A_{total} - (1 - \eta_f) N A_f}{A_{total}} = 1 - \frac{N A_f}{A_{total}} (1 - \eta_f) .$$

Eq. (3.63) can be used to calculate the total heat transfer for a fin array.

*IED Ex. 3.9
pp 144*

CHAPTER 4

Transient Conduction

4.1. Generalities

There are of course numerous engineering situations where unsteady effects are important. For example, the annealed steel quenching problem that we alluded to in Chapter 1. Consider also solar systems whose input is a function of the sun’s position, which changes as a function of time, or the start-up and shut-down of any generic thermal system, for instance internal combustion engines. Mathematically, these represent fairly complicated conduction problems, i.e. whereas steady 1-D conduction problems can be reduced to an ordinary differential equation, transient problems (in general) result in a partial differential equation, which is typically much more difficult to solve. Specifically, if we simplify the general equation of heat conduction, Eq. (2.18) on pp. 12, to model a 1-D transient problem, we get

$$(4.1) \quad \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

where α is the thermal diffusivity. This equation is quite a bit more difficult to solve than what we’ve seen up to now.

4.2. Lumped Capacitance Analysis

That said, we’ll now reverse ourselves and look at a special case where the problem can still be described by a fairly simple ordinary differential equation. Consider the conceptual case where spatial gradients in a material vanish, i.e. the temperature in the material is spatially uniform at any instant in time^{4.1}. By Fourier’s Law, it is implied that heat transfer with no gradient can only take place if conductivity approaches infinity, a physical situation that is not realistic within our current framework. Specifically

$$q'' = -k \frac{dT}{dx} \neq 0$$

if $dT/dx \rightarrow 0$, but $k \rightarrow \infty$. So although this situation is not actually possible, it may be closely approximated if the resistance to conduction within the material is low compared to the resistance of energy transfer from the boundary of the material to its surroundings. In other words, the approximation may be valid if the convection at the boundary is the “rate-determining

^{4.1}For example, recall that the temperature in the electrical wire problem back in Chapter 3 was very nearly constant.

step”. The method we derive to treat this configuration is *lumped capacitance analysis*. Let us assume, for the moment, that temperature in a conducting body is approximately constant. Gradients do not arise, so the conventional conduction equation is not relevant. Instead, we can write an overall energy balance, akin to Eq. (1.1) on pp. 3, except that we need only account for the energy transfer that takes place at the boundary of the conductor by the mechanism of convection. That is

$$\dot{E}_{stored} = -\dot{E}_{out},$$

where \dot{E}_{out} is the rate at which convection removes heat energy. This is once again quantified by Newton’s law of Cooling as $\dot{E}_{out} = hA_{surf}(T - T_{\infty})$. The rate at which heat energy stored decreases is $\dot{E}_{stored} = \rho V c dT/dt$. This gives

$$\rho V c \frac{dT}{dt} = -hA_{surf}(T - T_{\infty}).$$

Or, making the standard change of variables as introduced in Chapter 3, $\theta(t) = T(t) - T_{\infty}$, this becomes

$$(4.2) \quad \frac{d\theta}{dt} = -\frac{hA_{surf}}{\rho V c} \theta.$$

Eq. (4.2) is a so-called *separable differential equation*, which can be conveniently solved as follows. First, re-arrange Eq. (4.2) so the variables are “separated” as

$$\frac{d\theta}{\theta} = -\frac{hA_{surf}}{\rho V c} dt$$

then integrate over a finite time

$$\int_{\theta_0}^{\theta} \frac{d\theta'}{\theta'} = -\frac{hA_{surf}}{\rho V c} \int_0^t dt',$$

where t' and θ' are variables of integration. Note that on the left hand side, we are explicitly using limits that correspond to the initial condition (at $t = 0$) and to the condition at the time t of interest, i.e.

$$\theta|_{t'=0} = \theta_0 \quad \text{and} \quad \theta|_{t'=t} = \theta(t) = \theta.$$

Carrying out the integration, we find

$$\ln \theta' \Big|_{\theta_0}^{\theta} = -\frac{hA_{surf}}{\rho V c} t' \Big|_0^t,$$

which evaluates to

$$(\ln \theta - \ln \theta_0) = -\frac{hA_{surf}}{\rho V c} t.$$

Using the logarithm identity $(\ln a - \ln b) = \ln(a/b)$, we can write this expression more conveniently as

$$\ln \left(\frac{\theta}{\theta_0} \right) = -\frac{hA_{surf}}{\rho V c} t.$$

Finally, we can exponentiate both sides to obtain the more useful form

$$(4.3) \quad \theta = \theta_0 \exp\left(-\frac{hA_{surf}}{\rho Vc} t\right).$$

Substituting back the actual temperatures, we can also write the alternative, but equivalent form

$$(4.4) \quad \frac{T(t) - T_\infty}{T(0) - T_\infty} = \exp\left(-\frac{hA_{surf}}{\rho Vc} t\right),$$

where $T(0)$ is the initial temperature of the conductor.

According to Eq. (4.4), temperature response is a simple exponential when spatial gradients are small enough to be neglected. We can write this equation as

$$(4.5) \quad \frac{T(t) - T_\infty}{T(0) - T_\infty} = e^{-t/\tau},$$

where

$$(4.6) \quad \tau = \frac{\rho Vc}{hA_{surf}} = \frac{1}{hA_{surf}} (\rho Vc)$$

is a “time constant”. Notice that the first term is the standard form for the convective resistance we examined back in Chapter 3. Increasing this resistance, i.e. decreasing either h , A_{surf} , or both, increases the response time of the system. The second term, ρVc , is the *lumped thermal capacitance* of the conductor. Likewise, any increase of this term increases response time.

In engineering systems, we would be interested not only in $T(t)$, but also in how much heat was transferred over a particular time period. Determining the total energy transferred over some period of time is simply a matter of integrating Newton’s Law of Cooling over a particular time range

$$Q = \int_0^t q dt' = hA_{surf} \int_0^t \theta dt',$$

which gives

$$(4.7) \quad Q = \rho Vc \theta_0 (1 - e^{-t/\tau}).$$

4.3. Applicability of Lumped Capacitance

The lumped capacitance method provides a very convenient analysis of problems where gradients are negligible, with solutions for temperature and total heat transfer given respectively by Eqs. (4.3) and (4.7). Aside from the qualitative constraint about gradients, how do we actually determine whether such analysis is applicable? Recall, we know conceptually that this method is valid if the tendency for energy transfer via conduction within the material is much greater than the tendency for energy transfer via convection at the boundary. In other words, the resistance to conduction within the material must be low compared to the resistance of energy transfer from the boundary of the material to its surroundings. We can quantify this using a

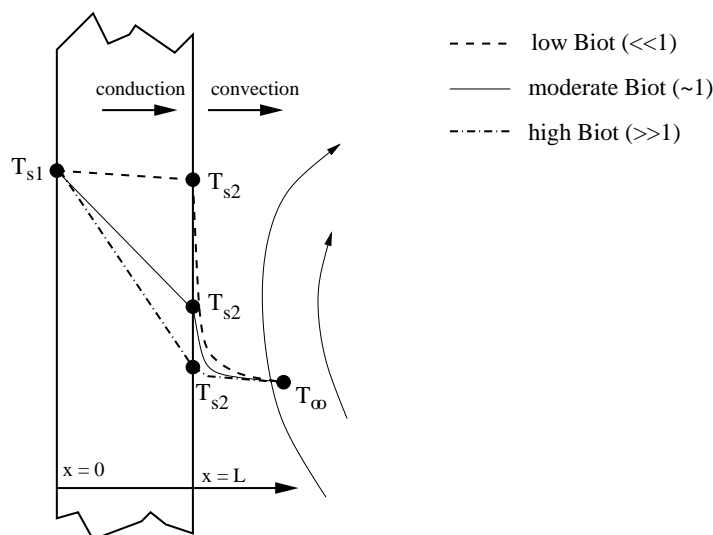


FIGURE 4.1. Diagram of relative magnitudes of temperature gradients for various Biot numbers.

simple conduction–convection energy equation written at a boundary, e.g. at $x = L$ in Fig. 4.1. Assuming no heat generation, we use a straight–line gradient to represent dT/dx in Fourier’s Law, so that we write

$$\frac{kA(T_{s1} - T_{s2})}{L} = hA(T_{s2} - T_{\infty}).$$

Re–arranging terms, we find

$$(4.8) \quad \frac{T_{s1} - T_{s2}}{T_{s2} - T_{\infty}} = \frac{hL}{k} = Bi,$$

where Bi is called the *Biot number*. We emphasize that k is the thermal conductivity of the conductor, not the surrounding fluid^{4.2}. The Biot Number is a dimensionless parameter that describes the ratio of the temperature drop (gradient) in the solid relative to the temperature drop (gradient) from the surface to the fluid. Recalling our assumption of small temperature gradient in the material, we would expect lumped capacitance analysis to be valid when the Biot Number is very small, i.e. $Bi \ll 1$. Notice further Eq. (4.8) provides a straightforward way to evaluate Bi based on measurable parameters of the problem without even having to know what these gradients are in advance. Mathematically, we can restate the simplification as $T(x, t) \rightarrow T(t)$ for $Bi \ll 1$. It follows that *the very first thing one should do when confronted with a transient temperature problem is check the Biot Number!* Convention has it that lumped capacitance can be used when

^{4.2}We will find a dimensionless expression similar to Eq. (4.8) called the Nusselt number arises in convection, except where k in that case is the conductivity of the surrounding fluid.

$Bi < 0.1$ (Incropera and Dewitt, 2002). Error in neglecting gradients is then less than 5 % (Özişik, 1985).

The values for k and h are clear, as is the value of L if the geometry is simple, e.g. as in Fig. 4.1. However, what is the length scale L if the geometry is more complicated? Often, we resort to using a so-called *characteristic length scale* L_c which can be defined as, for example, the ratio of the volume to the surface area *in the direction of heat transfer* of the conductor

$$L_c = \frac{V}{A_{surf}} .$$

For example, this convention can be applied to some of the most common geometries, e.g. as shown in Fig. 4.2.

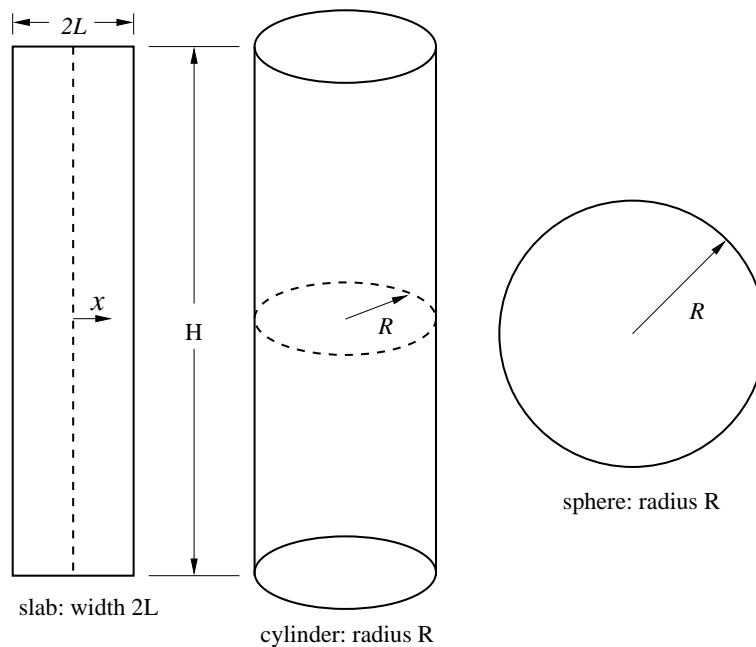


FIGURE 4.2. Common geometries for which characteristic length scales can be determined.

- Slab: $V = 2L \times H \times D$, where D denotes depth. Area is both sides in the direction of heat transfer: $A_{surf} = 2(H \times D)$, so that $L_c = L$.
- Cylinder: $V = \pi R^2 H$ and area is the surface area in the (radial) direction of heat transfer $2\pi R H$, so that $L_c = R/2$.
- Sphere: $V = 4\pi R^3/3$ and $A = 4\pi R^2$ giving $L_c = R/3$.

Specific problems may observe other conventions, for instance the length scale might be chosen to correspond to the dimension where the maximum ΔT occurs. This may be more conservative.

4.4. Casting the General One-Dimensional Problem

The condition $Bi \ll 1$ clearly excludes many cases of interest. For configurations where Lumped Capacitance analysis is not valid, we must go back to the governing equation of conduction, Eq. (4.1), which we recall is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

where α is the thermal diffusivity. This is a partial differential equation rather than an ordinary differential equation, and is therefore more difficult to solve. In general, we obtain the solution $T(x, t)$ according to the following procedure.

- Obtain the general solution via advanced calculus techniques, including separation of variables and integral transforms (Özişik, 1980).
- Specify 1 initial condition. This will be used to evaluate the constant of integration resulting from the first-order temporal term. We recall the basic form from Lumped Capacitance analysis, although, now instead of simply being constant, the initial temperature may be some function of the spatial variable

$$T(x, 0) = T_0(x).$$

- Specify 2 boundary conditions. These will be used to evaluate the 2 constants of integration resulting from the second-order spatial term. We recall this from steady 1-D conduction analysis, for example an insulated surface at the left and a convective boundary condition for the surface on the right:

$$\left. \frac{dT}{dx} \right|_{x=0} = 0$$

and

$$h[T(L, t) - T_\infty] = -k \left. \frac{dT}{dx} \right|_{x=L}.$$

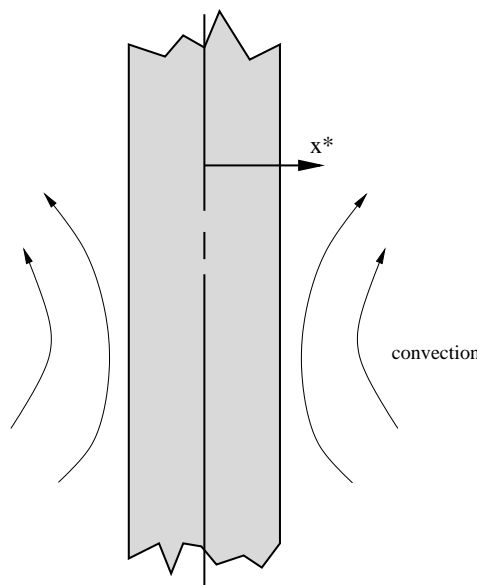
Therefore, the problem is a multivariate function

$$(4.9) \quad T = T(x, t, T_0, T_\infty, L, k, \alpha, h).$$

The physical manifestation of this problem is shown in Fig. 4.3. The “insulated” boundary condition is equivalent to a “symmetry” condition at $x = 0$. That is, no heat is conducted across the $x = 0$ boundary, so the temperature profile is symmetric about this point.

Before delving into this problem, it is worthwhile to mention the standard practice of non-dimensionalizing the system, which was probably covered in fluid mechanics^{4.3}. One of the primary advantages of this process is to reduce the number of relevant variables in the problem. For example,

^{4.3}Panton (1984) has an extensive discussion of dimensional analysis, including proofs of the relevant theorems.

FIGURE 4.3. *Diagram of transient problem.*

recall how 3 separate variables in a standard flow problem: characteristic velocity and length, u_0 and L_0 , and kinematic viscosity, ν , can be combined into a single group, the Reynolds number, Re . In a typical study of drag, D , for an aerodynamic body, instead of plotting families of curves for u_0 , L_0 , and ν versus D , we simply plot Re versus D . So what are the relevant non-dimensional groups in our 1-D conduction problem? Going through the non-dimensionalization process using the Buckingham-II Theorem, we find a number of relevant “groups”:

- Dimensionless temperature: According to our now-standard variable substitution $\theta = T - T_\infty$, we can write a dimensionless temperature as

$$\theta^* = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{\theta}{\theta_0},$$

which clearly implies the limits $0 \leq \theta^* \leq 1$.

- Dimensionless spatial coordinate: Using the characteristic length dimension, we obtain

$$x^* = \frac{x}{L}.$$

- Dimensionless time coordinate: Time may be non-dimensionalized using the characteristic length dimension and the thermal diffusivity, as in

$$t^* = \frac{\alpha t}{L^2} = Fo,$$

where Fo is commonly called the *Fourier Number*^{4.4}.

To apply the same dimensionless treatment to the governing equation, Eq. (4.1), we first determine how derivatives behave in the dimensionless realm using the Chain Rule of Calculus, as in

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*} \frac{\partial t^*}{\partial t} = \frac{\alpha}{L^2} \frac{\partial}{\partial t^*}$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x^*} \frac{\partial x^*}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x^*}.$$

The latter expression leads us to the required second derivative in x , as in

$$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{1}{L} \frac{\partial}{\partial x^*} \right) = \frac{1}{L} \frac{\partial}{\partial x^*} \frac{\partial x^*}{\partial x} \left(\frac{\partial}{\partial x^*} \right) = \frac{1}{L^2} \frac{\partial^2}{\partial x^{*2}}.$$

Using these expressions, Eq. (4.1) can be written in the dimensionless form

$$(4.10) \quad \frac{\partial \theta^*}{\partial t^*} = \frac{\partial^2 \theta^*}{\partial x^{*2}}$$

Likewise, the initial condition becomes

$$(4.11) \quad \theta^*(x^*, 0) = 1$$

and boundary conditions become

$$(4.12) \quad \left. \frac{d\theta^*}{dx^*} \right|_{x^*=0} = 0$$

and

$$(4.13) \quad -Bi \cdot \theta^*(1, t^*) = \left. \frac{d\theta^*}{dx^*} \right|_{x^*=1}.$$

The number of relevant variables has been dramatically reduced as compared to the dimensional form of the problem stated in Eq. (4.9). That is, the problem is now

$$(4.14) \quad \theta^* = \theta^*(x^*, t^*, Bi),$$

where the single parameter, the Biot number, arises in the boundary conditions.

^{4.4}This is the unfortunate nomenclature used in most texts. The Fourier number is not a dimensionless “number” in the same sense as, for example the Reynolds number in fluid mechanics. It is a *function* of time, although it does not have any units. Therefore, it would be more appropriately referred to as a dimensionless coordinate. However, as a first approximation, it does also connote the relative effectiveness with which a body conducts and stores energy (e.g. Incropera and Dewitt, 2002, pp. 256). In that sense, it could also be loosely interpreted as a “dimensionless number”.

4.5. Transient Analysis in One Dimension

Eqs. (4.10) through (4.13) represent a linear partial differential system. As mentioned, the solution process involves advanced techniques, including separation of variables or integral transforms (Özişik, 1980). At this point, rather than examining details of the actual solution process for this system, let us use the well-established solution directly. The solution takes the form of an infinite series

$$(4.15) \quad \theta^* = \sum_{n=1}^{\infty} C_n e^{-\zeta_n^2 t^*} \cos \zeta_n x^* ,$$

where the mode coefficients are

$$(4.16) \quad C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)}$$

and ζ_n are *eigenvalues* given by the roots of the transcendental equation

$$(4.17) \quad \zeta_n \tan \zeta_n = Bi .$$

Several roots for various values of Bi have been tabulated (e.g. Incropera and Dewitt, 2002, Appendix B.3, pp. 936).

This solution is quite a bit more complex than what we have dealt with so far, especially since the eigenvalues are not given by an explicit relationship. However, except for small values of the Fourier Number ($t^* < 0.2$), the series can be approximated by the first term only. Thus, for $t^* = Fo > 0.2$, the solution can be taken to be

$$(4.18) \quad \theta^* = C_1 e^{-\zeta_1^2 t^*} \cos \zeta_1 x^* ,$$

where

$$(4.19) \quad C_1 = \frac{4 \sin \zeta_1}{2\zeta_1 + \sin(2\zeta_1)} .$$

We can express the solution equivalently as

$$(4.20) \quad \theta^* = \theta_0^* \cos \zeta_1 x^* ,$$

where θ_0^* is the time-dependent temperature at the mid-plane $x = 0$

$$(4.21) \quad \theta_0^* = C_1 e^{-\zeta_1^2 t^*} .$$

A condensed summary of coefficients C_1 and the eigenvalues ζ_1 is provided in Table 4.1. Extensive tabulations are available in reference texts (e.g. Incropera and Dewitt, 2002, pp. 258).

To calculate the total amount of heat transferred over some interval, we again resort to our basic conservation equation

$$\Delta E_{stored} = E_{in} - E_{out} ,$$

where $Q = E_{out}$, $E_{in} = 0$, and $\Delta E_{stored} = E(t) - E(0)$. Substituting these quantities, we can integrate the expression over the domain to obtain the

TABLE 4.1. Coefficients and eigenvalues for the one-term approximate solution of transient heat conduction in the rectangular coordinate system.

Bi	ζ_1	C_1	Bi	ζ_1	C_1	Bi	ζ_1	C_1
0.01	0.0998	1.0017	0.2	0.4328	1.0311	2.0	1.0769	1.1795
0.02	0.1410	1.0033	0.25	0.4801	1.0382	3.0	1.1925	1.2102
0.03	0.1732	1.0049	0.3	0.5218	1.0450	4.0	1.2646	1.2287
0.04	0.1987	1.0066	0.4	0.5932	1.0580	5.0	1.3138	1.2402
0.05	0.2217	1.0082	0.5	0.6533	1.0701	6.0	1.3496	1.2479
0.06	0.2425	1.0098	0.6	0.7051	1.0814	7.0	1.3766	1.2532
0.07	0.2615	1.0114	0.7	0.7506	1.0919	8.0	1.3978	1.2570
0.08	0.2791	1.0130	0.8	0.7910	1.1016	9.0	1.4149	1.2598
0.09	0.2956	1.0145	0.9	0.8274	1.1107	10.0	1.4289	1.2620
0.10	0.3111	1.0160	1.0	0.8603	1.1191	20.0	1.4961	1.2699
						∞	1.5707	1.2733

total heat transfer, as in

$$Q = -[E(t) - E(0)] = - \int_{vol} \rho c [T(t) - T_0] dV .$$

Let us introduce a (non-obvious) factor Q_0 to non-dimensionalize this equation based on the initial temperature difference, i.e. the temperature difference at $t = 0$

$$Q_0 = \rho c V [T_0 - T_\infty] .$$

The expression Q_0 is not only the initial energy present, it is also the maximum amount of energy which could be transferred as $t \rightarrow \infty$. The ratio Q/Q_0 is therefore the fraction of total energy transferred over an interval $0 \rightarrow t$. We find

$$\frac{Q}{Q_0} = \int_{vol} \frac{-\rho c [T(t) - T_0]}{\rho c [T_0 - T_\infty]} \frac{dV}{V} = \int_{vol} \frac{-[T(t) - T_0]}{[T_0 - T_\infty]} \frac{dV}{V} ,$$

which can be manipulated further by adding and subtracting T_∞ in the numerator to obtain

$$\frac{Q}{Q_0} = - \int_{vol} \left(\frac{T(t) - T_\infty}{T_0 - T_\infty} - \frac{T_0 - T_\infty}{T_0 - T_\infty} \right) \frac{dV}{V} .$$

Using our definition of θ^* , this simplifies to

$$\frac{Q}{Q_0} = \frac{1}{V} \int_{vol} (1 - \theta^*) dV .$$

Integration can be carried out by substituting the one-term approximate solution, preferably Eq. (4.20), into this expression. Recall, the length of our dimensionless domain varies as $0 \leq x^* \leq 1$ from the mid-plane to the

edge. If we take w^* and l^* to be dimensionless width and depth of our domain, we have $dV = w^*l^*dx^*$ and $V = w^*l^* \cdot 1 = w^*l^*$. We thus find

$$\frac{Q}{Q_0} = \frac{1}{w^*l^*} \int_0^1 (1 - \theta_0^* \cos \zeta_1 x^*) w^*l^* dx^* = \int_0^1 (1 - \theta_0^* \cos \zeta_1 x^*) dx^* .$$

Again, recall that θ_0^* only describes the time-dependent temperature behavior at the mid-plane, so it is not a function of dx^* . Also, the eigenvalue ζ_1 is a constant. We therefore have a simple “cosine” integration, which yields

$$\frac{Q}{Q_0} = \left[x^* - \frac{\theta_0^*}{\zeta_1} \sin \zeta_1 x^* \right] \Big|_0^1 ,$$

which evaluates to

$$(4.22) \quad \frac{Q}{Q_0} = 1 - \frac{\theta_0^*}{\zeta_1} \sin \zeta_1 .$$

These results can, of course, also be applied to a slab of thickness L with an insulated boundary on the left at $x^* = 0$, since the mathematical problem, specifically the boundary conditions, are exactly the same as the problem we have just examined. Also, our result can also be applied to problems where there is a step change of the surface temperature on free boundary at $x^* = 1$ to a new value T_s . Conceptually, we would realize the same result if we allowed a step change in fluid temperature $T_\infty \rightarrow T_s$ and prescribed an infinite Biot number^{4.5}, so that the change in fluid temperature would also be immediately transmitted to the surface at $x^* = 1$. Thus, step change problems of this sort are handled by allowing $Bi \rightarrow \infty$.

IED Ex. 5.4
pp 262

For other systems, e.g. the infinitely long cylinder and the sphere, similar approaches can be taken to determine exact solutions. These are even more complicated than slab problems because of the additional considerations that non-rectangular coordinate systems place on the problem. For example, the solution for radial one-dimensional transient conduction is

$$\theta^* = \sum_{n=1}^{\infty} \left(\frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)} \right) e^{-\zeta_n^2 t^*} J_0(\zeta_n r^*) ,$$

where $t^* = \alpha t / r_0^2$ and J_0 and J_1 are Bessel functions of the first kind. Here, eigenvalues are the roots of

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi .$$

So, we find that the problems are more difficult, but their solutions appear to be more difficult to evaluate, as well. Luckily, we find that the one-term approximation is again valid for $t^* = Fo > 0.2$, so that tabulated values (e.g. Incropera and Dewitt, 2002, pp. 258) can be used directly.

^{4.5}Recall from Eq. (4.8) the form of the Biot number is $Bi = hL/k$, so that an infinite Biot number is equivalent to prescribing an infinite convection coefficient h .

4.6. The Similarity Technique

Another geometry which can be studied analytically is that of a semi-infinite solid, which serves as a useful idealization for many problems, e.g. any body which is very “thick” Fig. 4.4. This sort of problem is described

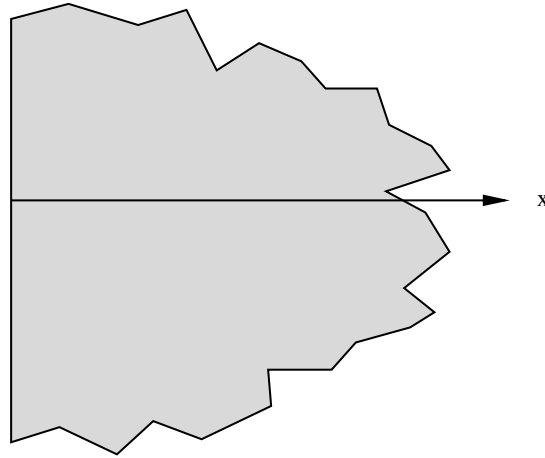


FIGURE 4.4. *Diagram of semi-infinite domain.*

by the same equation and initial conditions as the other problems, however, boundary conditions are special. We still require 2 boundary conditions because of the second-order spatial derivative, but there is clearly only 1 surface to specify a boundary condition at, i.e. $x = 0$. There is no boundary at $x \rightarrow \infty$. However, we can use the conceptual boundary condition of

$$T(x \rightarrow \infty, t) = T_0,$$

which says simply that for some value x that is large enough, the temperature will always be the initial temperature. Physically, this makes sense because if truly $x \rightarrow \infty$, then we can always pick x large enough such that it is so far away from the boundary at $x = 0$ that temperature variations cannot be perceived. For the surface, we can specify any of the 3 standard types of boundary conditions we’ve studied.

The mathematical significance of this problem is that there is a very clever trick to simplify the partial differential equation, Eq. (4.1), to an ordinary differential equation. This trick is known as a *similarity transform* and effectively converts the problem of two independent variables, x and t , into a problem of just one independent variable, the similarity transform variable η . Such a transform is often seen in fluid mechanics and may be possible for any problem that doesn’t have naturally-identifiable scales, for example problems where there’s no clear length scale because of infinite dimensions. Determining what the similarity transform variable is for a

specific problem is not trivial! In this case, the correct variable is

$$\eta = \frac{x}{\sqrt{4\alpha t}}.$$

Let us apply this to Eq. (4.1).

First, we use the Chain Rule to determine what the derivatives are with respect to the new (transformed) variable η , specifically

$$\frac{\partial T}{\partial t} = \frac{dT}{d\eta} \frac{\partial \eta}{\partial t} = -\frac{2x\alpha}{(4\alpha t)^{3/2}} \frac{4\alpha t}{4\alpha t} \frac{dT}{d\eta} = -\frac{x}{2t\sqrt{4\alpha t}} \frac{dT}{d\eta}$$

and

$$\frac{\partial T}{\partial x} = \frac{dT}{d\eta} \frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}} \frac{dT}{d\eta}.$$

Another derivative in x yields

$$\frac{\partial^2 T}{\partial x^2} = \frac{d}{d\eta} \left(\frac{\partial T}{\partial x} \right) \frac{\partial \eta}{\partial x} = \frac{d}{d\eta} \left(\frac{1}{\sqrt{4\alpha t}} \frac{dT}{d\eta} \right) \frac{1}{\sqrt{4\alpha t}} = \frac{1}{4\alpha t} \frac{d^2 T}{d\eta^2}.$$

Substituting these into Eq. (4.1), we find

$$-\frac{x}{2t\sqrt{4\alpha t}} \frac{dT}{d\eta} = \alpha \frac{1}{4\alpha t} \frac{d^2 T}{d\eta^2},$$

which can be simplified to

$$\frac{d^2 T}{d\eta^2} = -\frac{4xt}{2t\sqrt{4\alpha t}} \frac{dT}{d\eta},$$

and finally

$$(4.23) \quad \frac{d^2 T}{d\eta^2} = -2\eta \frac{dT}{d\eta}.$$

Note that Eq. (4.23) is now an ordinary differential equation (second-order), having the single independent variable η .

The most straightforward method to solve this equation is to make the variable substitution

$$\xi = \frac{dT}{d\eta},$$

so that Eq. (4.23) is written as

$$\frac{d}{d\eta} \left(\frac{dT}{d\eta} \right) = -2\eta \frac{dT}{d\eta}$$

and recast according to the new variable as

$$(4.24) \quad \frac{d\xi}{d\eta} = -2\eta \xi.$$

Eq. (4.24) is separable, so that it can be written as

$$\frac{d\xi}{\xi} = -2\eta d\eta$$

and then integrated once to obtain $\ln \xi = -\eta^2 + C_0$, where C_0 is an integration constant. We can write an equivalent equation in exponential form as

$$\xi = e^{-\eta^2 + C_0} = C_1 e^{-\eta^2},$$

where C_1 replaces the equivalent constant $\exp(C_0)$. Recalling our variable substitution, we can now write this equation in the form

$$(4.25) \quad \frac{dT}{d\eta} = C_1 e^{-\eta^2}.$$

Eq. (4.25) is the result of doing one integration on our second-order Eq. (4.23). While this equation appears to be in good shape to perform the second integration, i.e.

$$(4.26) \quad T = C_1 \int e^{-\eta^2} d\eta + C_2,$$

the process is actually not trivial. If, for example, we prescribe boundary conditions of the first-kind, e.g. $T = T_s$ at $x = 0$, we obtain the solution

$$(4.27) \quad \frac{T - T_s}{T_0 - T_s} = \operatorname{erf} \eta,$$

where erf is the *Gaussian Error Function*. This function is tabulated in reference texts (e.g. Incropera and Dewitt, 2002, Appendix B.2, pp. 935). The surface heat flux can be obtained by applying Fourier's Law, from which one obtains

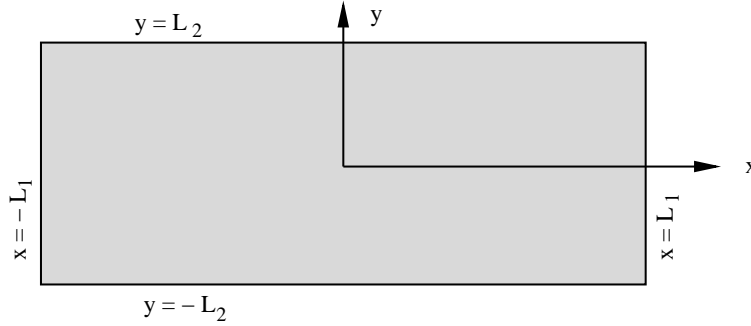
$$(4.28) \quad q_s'' = k \frac{T_s - T_0}{\sqrt{\pi \alpha t}}.$$

Although somewhat more involved, solutions for boundary conditions of the second and third kind can be obtained as well (Incropera and Dewitt, 2002).

4.7. Multi-dimensional Transient Conduction

Transient problems that depend upon more than 1 dimension are often encountered in practice. If there is no energy generation, it is possible to combine 1-D solutions that we have looked at to construct, for example, 2-D solutions for transient problems. In particular, we can show that if the one-term approximation is valid, then multi-dimensional solutions are merely the simple products of corresponding one-dimensional solutions. This approach is called the method of *product solution*, and is applicable only if the solution of the general problem can be shown to be equivalent to the product of 2 one-dimensional transient heat conduction problems. What is the theory behind this?

Consider a rectangular region of dimension $2L_1 \times 2L_2$, where $-L_1 \leq x \leq L_1$ and $-L_2 \leq y \leq L_2$ and where the initial temperature is T_0 (Fig. 4.5). The

FIGURE 4.5. *Two-dimensional heat conduction domain.*

problem is driven by convection on all 4 faces at an ambient temperature of T_∞ . The dimensionless temperature is defined as

$$\theta(x, y, t) = \frac{T(x, y, t) - T_\infty}{T_0 - T_\infty},$$

where $\theta(x, y, t)$ is governed by the multi-dimensional equation

$$(4.29) \quad \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}.$$

Eq. (4.29) is obtained via standard non-dimensionless techniques^{4.6} shown in § 4.4. We will examine the product solution technique for the governing equation in 4.29 with the realization that boundary conditions are treated in a similar fashion.

Let us now assume that we can represent $\theta(x, y, t)$ in the form of the product $\theta(x, y, t) = \theta_1(x, t) \cdot \theta_2(y, t)$. The components $\theta_1(x, t)$ and $\theta_2(y, t)$ are one-dimensional solutions in the x and y coordinate directions, respectively, and are further assumed to be given by the one-dimensional approximation. That is, we assume these functions satisfy

$$(4.30) \quad \frac{\partial \theta_1}{\partial t} = \frac{\partial^2 \theta_1}{\partial x^2}$$

and

$$(4.31) \quad \frac{\partial \theta_2}{\partial t} = \frac{\partial^2 \theta_2}{\partial y^2}$$

in the x and y directions^{4.7}. Now, substitute $\theta(x, y, t) = \theta_1(x, t) \cdot \theta_2(y, t)$ into Eq. (4.29). For various components we obtain

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta_1 \cdot \theta_2}{\partial t} = \theta_2 \frac{\partial \theta_1}{\partial t} + \theta_1 \frac{\partial \theta_2}{\partial t}$$

^{4.6}We have omitted the star notation indicating dimensionless variables as shown in § 4.4 for brevity. Independent variables in Eq. (4.29) are taken to be dimensionless.

^{4.7}Compare these equations to Eq. (4.10).

and

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta_1 \cdot \theta_2}{\partial x^2} = \frac{\partial}{\partial x} \left(\theta_2 \frac{\partial \theta_1}{\partial x} + \theta_1 \frac{\partial \theta_2}{\partial x} \right),$$

so that

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta_2}{\partial x} \frac{\partial \theta_1}{\partial x} + \theta_2 \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial \theta_1}{\partial x} \frac{\partial \theta_2}{\partial x} + \theta_1 \frac{\partial^2 \theta_2}{\partial x^2}.$$

However, recalling that $\partial \theta_2 / \partial x = 0$ since θ_2 is not a function of the x coordinate, this term simplifies to

$$\frac{\partial^2 \theta}{\partial x^2} = \theta_2 \frac{\partial^2 \theta_1}{\partial x^2}.$$

According to a similar procedure, we see that

$$\frac{\partial^2 \theta}{\partial y^2} = \theta_1 \frac{\partial^2 \theta_2}{\partial y^2}.$$

Substituting these terms into Eq. (4.29) and moving everything to the left hand side, we get

$$(4.32) \quad \theta_2 \left(\frac{\partial \theta_1}{\partial t} - \frac{\partial^2 \theta_1}{\partial x^2} \right) + \theta_1 \left(\frac{\partial \theta_2}{\partial t} - \frac{\partial^2 \theta_2}{\partial y^2} \right) = 0.$$

Eq. (4.32) is the original governing equation written in terms of our product solution hypothesis. If it is valid, then either θ_1 and θ_2 must both vanish, or the terms within the brackets must both vanish. Now, θ_1 and θ_2 do not generally vanish, otherwise the problem is trivial. Examining this equation more closely we see that the terms in brackets are nothing more than Eqs. (4.30) and (4.31). Thus, the terms in brackets do in fact vanish, and the original hypothesis of the product solution is valid. As mentioned above, boundary conditions can be treated similarly. Note how the presence of a heat generation term would prevent us from drawing our final conclusion since neither of the equations for θ_1 and θ_2 would necessarily be zero. Of course, from our one-dimensional work, we know various solutions corresponding to θ_1 and θ_2 and the rules under which they (and their simplifications) can be applied. So, once again, we've managed to find a significant simplification for a special case, i.e. two-dimensional problem with no heat generation.

IED Ex. 5.7
pp 277

Once it is determined that the product solution is valid, solving a problem proceeds by treating the one-dimensional problems individually. In particular

- calculate individual Biot numbers for each coordinate direction using the appropriate length scale for each coordinate
- likewise, calculate individual Fourier numbers (dimensionless times) for each coordinate direction
- compute individual one-dimensional solutions, e.g. θ_1 and θ_2 , for the required conditions of the two-dimensional problem
- take the product of the one-dimensional expressions as the final solution of the problem

CHAPTER 5

Introduction to Convection

So far we have only considered convection to the extent that it supplies a boundary condition for conduction problems. However, in many cases, convection, driven by bulk motion of the participating medium, is the dominant mode of heat transfer, even though there remains some level of conduction. In this chapter, we focus on the underlying physics before actually developing calculation methods for quantifying the heat transfer.

Consider a fluid moving over a flat plate at free-stream velocity u_∞ and free-stream temperature T_∞ , where the surface of the plate is at temperature T_s (Fig. 5.1). If $T_s \neq T_\infty$ then we can write the heat transfer for a differential

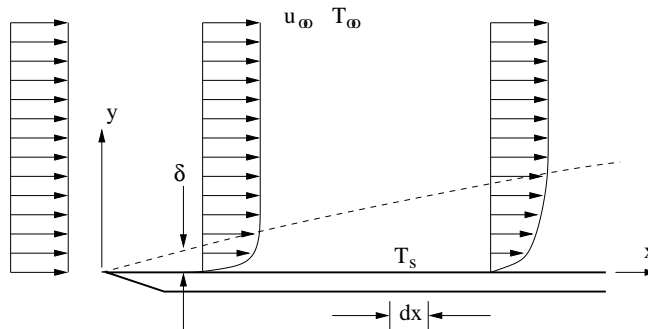


FIGURE 5.1. Flat-plate configuration showing boundary layer.

surface area using the usual Newton's Law of Cooling

$$dq = h (T_s - T_\infty) dA_s ,$$

where h is formally defined as the local coefficient of heat transfer or the local convection coefficient. Whereas before we assume that h is a constant, we know that, in reality, h will vary from point to point on the surface because the flow conditions will vary. Clearly, the total heat transfer can be obtained by integrating over the surface, as in

$$q = \int_{A_s} h (T_s - T_\infty) dA_s .$$

One of our basic assumptions will be that both T_∞ and T_s are constant, so that their difference, $T_s - T_\infty$, is also constant. The integration can then be

simplified as

$$q = (T_s - T_\infty) \int_{A_s} h dA_s .$$

Let us separately define an *average* heat transfer coefficient by again applying Newton's Law of Cooling

$$q = \bar{h} (T_s - T_\infty) A_s .$$

From these two equations, it is clear that we can write the average convection coefficient in terms of its local value as

$$(5.1) \quad \bar{h} = \frac{1}{A_s} \int_{A_s} h dA_s .$$

So we see that what we were really talking about all along in conduction was an averaged value of the convection coefficient. Now the task becomes actually figuring out what this is for specific problems.

For the special case of the flat plate in Fig. 5.1, the average convection coefficient can be simplified. If we take the length of the plate as L and the width as W , then $A_s = W \times L$ and $dA_s = W \times dx$. Integrating over the length of the plate from the leading edge $0 \rightarrow L$, we find

$$(5.2) \quad \bar{h} = \frac{1}{L} \int_0^L h dx .$$

Determining convection coefficients is viewed as the "convection problem". However, the problem is not trivial, since now we have the additional complexity of fluid motion. So, not only is geometry, specific heat, conductivity, etc. important, we must now also worry about flow conditions, fluid density, viscosity, etc.. Much of this depends upon the concept of boundary layer theory, which can be used to describe flow regions in the neighborhood of a solid surface.

IED Ex. 6.1
pp 329

5.1. Boundary Layer Introduction

Recall the concept of the boundary layer from fluid mechanics. The drag imposed by the fixed no-slip surface acts through the mechanism of shear stress, τ , which gradually decreases as distance y from the plate increases. At a distance defined as $y = \delta$ the drag effect becomes negligible. Therefore, the velocity along the surface, u , gradually increases from zero at the surface (no-slip boundary conditions) to a value of u_∞ , the free-stream velocity. The convention used to define the boundary layer is $\delta = y$ at which $u = 0.99u_\infty$. Thus the flow problem is defined by two distinct regions:

- The boundary layer in which velocity gradients and shear stresses are significant.
- The region outside the boundary layer in which gradients and shear stresses are negligible.

These observations refer to the fluid problem, so this boundary layer is termed the *velocity boundary layer*. The primary factor, τ , is defined for a Newtonian fluid according to Newton's Law of Viscosity

$$\tau = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0}$$

at the plate's surface, where μ is the material property known as the dynamic viscosity.

Just as the velocity boundary layer develops for the fluid problem, a *thermal boundary layer* develops if the fluid free stream and the surface temperatures differ (Fig. 5.2). Specifically, assume $T_s > T_\infty$. The incoming

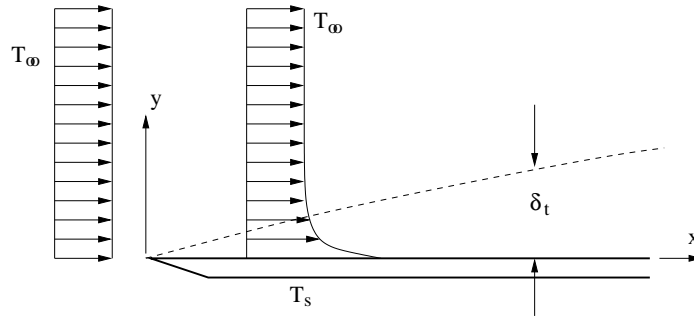


FIGURE 5.2. *Thermal boundary layer on a flat plate.*

flow has a uniform temperature distribution T_∞ , however, particles at the plate surface attain thermal equilibrium at the plate's higher surface temperature T_s . In turn, these particles exchange energy with those near them, etc., which is the same basic dynamic process familiar from the action of viscosity. Therefore, a temperature profile develops, in which temperature gradients within the thermal boundary layer, δ_t , are significant, and outside they are negligible. The layer itself is defined as y at which

$$\frac{T_s - T}{T_s - T_\infty} = 0.99 .$$

Note the correspondence with the factor of 0.99 for the velocity boundary layer. The behavior of the thermal boundary layer is qualitatively similar to the velocity boundary layer in that the thickness grows as x increases.

At the plate's surface, the no-slip condition indicates there is no fluid motion. It follows that heat transfer between the plate and the fluid immediately adjacent to it must result solely from conduction. That is, we can write heat transfer directly from Fourier's Law of Conduction as

$$q'' = -k_f \left. \frac{\partial T}{\partial y} \right|_{y=0} ,$$

where k_f is the thermal conductivity of the fluid. By combining this with a local heat flux derived from Newton's Law of Cooling, $q'' = h(T_s - T_\infty)$, we see

$$(5.3) \quad h = - \frac{k_f}{T_s - T_\infty} \left. \frac{\partial T}{\partial y} \right|_{y=0}.$$

From this, we make the following interesting observation: $T_s - T_\infty$ and k_f are constants, therefore variation in h depends upon $\partial T/\partial y$. Because δ_t increases with x , the gradient decreases with x , therefore h and q'' decrease with x . The integrals in Eqs. (5.1) and (5.2) are expected to be non-trivial since the local value of h in the integrand is not constant.

From fluid mechanics, we should recall that the physical structure of the boundary layer depends upon whether it is laminar or turbulent. The laminar regime is characterized by orderly flow in which streamlines can be easily identified, while turbulent flow is chaotic and irregular. Chaotic mixing in turbulent boundary layers increases momentum and energy transfer, so that convection and shear stress near the plate are generally higher than in the laminar case. The turbulent velocity profile is thus sharper than its laminar counterpart (Fig. 5.3). The relevant parameter for boundary layers

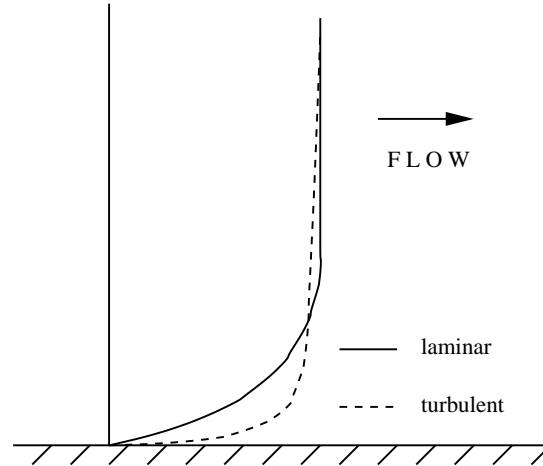


FIGURE 5.3. Qualitative characterization of velocity profiles in laminar and turbulent boundary layers.

is the *Reynolds number*, defined as

$$(5.4) \quad Re_x = \frac{u_\infty x}{\nu},$$

where x is distance traveled as measured from the leading edge, ν is the fluid kinematic viscosity, and u_∞ is the free-stream velocity of the flow. Transition from a laminar to a turbulent boundary layer occurs at approximately $Re_x \sim 5 \times 10^5$.

5.2. Governing Equations

From fluid mechanics, we recall further that the flow physics are governed by the *Navier–Stokes Equations*, which describe conservation of mass and conservation of momentum. If we assume steady flow over the flat plate in the (x, y) domain as shown in Figs 5.1 and 5.2, we can write these equations as

$$(5.5) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$(5.6) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

and

$$(5.7) \quad u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),$$

where (u, v) are the velocity components in the (x, y) coordinate directions, respectively. Eq. (5.5) represents the conservation of mass, while Eqs. (5.6) and (5.7) are conservation of momentum in the (x, y) coordinate directions, respectively.

These equations are typically derived using a differential approach much like that introduced in Chapter 2 for the conduction equation^{5.1}. Expanding this approach for thermal energy to include not only conduction, but also convection and the mechanisms which degrade mechanical energy into heat, we can derive the following equation for the conservation of energy

$$(5.8) \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \frac{\nu}{c_p} \Phi,$$

where

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2$$

is the viscous energy dissipation function. The left hand side of Eq. (5.8) represents the net outflow of energy from the control volume due to convection^{5.2}, while the term in brackets is the net rate at which energy is conducted into the control volume^{5.3}. The last term, Φ , denotes thermal energy that arises via the action of viscous (frictional) forces degrading mechanical energy.

This system of equations is, in fact, more comprehensive than what is actually necessary to model flow over the flat plate because certain gradients in the streamwise direction are very small compared to those in the normal direction. We can invoke the so-called *boundary layer approximation* to

^{5.1}Recall that Eqs. (5.6) and (5.7) arise from applying Newton's Second Law.

^{5.2}That is, the energy convected into the control volume minus the energy convected out.

^{5.3}Recall this interpretation from Eq. (2.19) on pp. 12.

derive a simplified system. The simplification process proceeds by examining the order of magnitude of each of the terms (Schlichting, 1979).

- Assume in continuity Eq. (5.5) that $\partial u/\partial x$ is of order unity, which we write as $\partial u/\partial x \sim [1]$. Since there are no parameters in this equation and we know that the flow must develop along the axis, i.e. $\partial/\partial x \neq 0$, the term $\partial v/\partial y$ must be the same order of magnitude. That is $\partial v/\partial y \sim [1]$.
- The free-stream flow is of order unity, i.e. $u \sim [1]$.
- The flow remains predominantly oriented along the plate so that the vertical velocity component is much smaller than the horizontal component, $v \ll u$. Therefore, we write $v \sim [\epsilon]$, where $\epsilon \ll 1$. That is, ϵ is a small number.
- Since changes in the vertical direction happen over a very short distance, i.e. the boundary layer thickness, then gradients in the vertical direction must be much larger than gradients along the plate, i.e. $\partial/\partial x \ll \partial/\partial y$. Since $u \sim [1]$ and $\partial u/\partial x \sim [1]$, we can say that $\partial/\partial x \sim [1]$. Since $\partial/\partial x \ll \partial/\partial y$, we can write $\partial/\partial y \sim [1/\epsilon]$. Note that we could also infer this from the fact that $\partial v/\partial y \sim [1]$, but that $v \sim [\epsilon]$. Thus, we see $\partial^2 u/\partial^2 x \sim [1]$, $\partial v/\partial x \sim [\epsilon]$ and $\partial^2 v/\partial^2 x \sim [\epsilon]$. Moreover, $\partial u/\partial y \sim [1/\epsilon]$, $\partial^2 u/\partial^2 y \sim [1/\epsilon^2]$, and $\partial^2 v/\partial^2 y \sim [1/\epsilon]$.
- In the boundary layer, inertial and viscous effects must be of the same overall order of magnitude. If we were to rewrite the Eqs. (5.5) through (5.7) in dimensionless form, i.e. using the Reynolds number as defined in Eq. (5.4), we would see that $Re \sim [1/\epsilon^2]$ for this to be the case.

Following the last point, let us rewrite Eqs. (5.5) through (5.7) in dimensionless form along with the orders of magnitude for each term we have deduced above. Showing the orders of magnitude in square brackets beside each term, we find

$$(5.9) \quad u \frac{\partial u}{\partial x} [1] \cdot [1] + v \frac{\partial u}{\partial y} [\epsilon] \cdot [\epsilon^{-1}] = -\frac{\partial P}{\partial x} + \frac{1}{Re} [\epsilon^2] \left(\frac{\partial^2 u}{\partial x^2} [1] + \frac{\partial^2 u}{\partial y^2} [\epsilon^{-2}] \right)$$

and

$$(5.10) \quad u \frac{\partial v}{\partial x} [1] \cdot [\epsilon] + v \frac{\partial v}{\partial y} [\epsilon] \cdot [1] = -\frac{\partial P}{\partial y} + \frac{1}{Re} [\epsilon^2] \left(\frac{\partial^2 v}{\partial x^2} [\epsilon] + \frac{\partial^2 v}{\partial y^2} [\epsilon^{-1}] \right)$$

for the momentum equations. As we said above, the entire continuity equation is of order unity, i.e.

$$(5.11) \quad \frac{\partial u}{\partial x} [1] + \frac{\partial v}{\partial y} [1] = 0.$$

We did not write down the orders of magnitude for the pressure terms in Eqs. (5.9) and (5.10). We can infer that the pressure gradient in Eq. (5.10) should be the same order of magnitude as the equation itself. In other

words, changes in pressure across the boundary layer should be of comparable size to the other terms in the equation. Specifically, $\partial P/\partial y \sim [\epsilon]$, since Eq. (5.10) is of overall order $[\epsilon]$. This leads us to conclude $P \sim [\epsilon^2]$ across the boundary layer (in the vertical direction), since $\partial P/\partial y \sim [\epsilon]$ and $\partial/\partial y \sim [1/\epsilon]$. Therefore, P is approximately only a function of x , so that we assume the pressure term in Eq. (5.9) is of the same overall magnitude as the equation itself, i.e. order $[1]$.

We can now make some simplifying observations. First, $\partial^2 u/\partial x^2$ in Eq. (5.9) is *very* small compared to the other terms in that equation, so it can be dropped. Also, the entire Eq. (5.10) is small compared Eq. (5.9), so that it may be neglected. These deductions show that Eq. (5.9) through (5.11) can be simplified to

$$(5.12) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2}$$

and

$$(5.13) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

We can also apply the same order of magnitude analysis for energy in Eq. (5.8). We assume

- temperature is of order unity, $T \sim [1]$
- similar to the momentum equations, we see that the convective terms will be of order unity, so that the viscous terms should also be of overall order unity — since the largest viscous term is $\partial^2 T/\partial^2 y \sim [1/\epsilon^2]$ (similar again to the momentum equations), we must have $\alpha \sim [\epsilon^2]$ for the overall viscous order to be unity.
- the dissipation term Φ should also be of overall order of magnitude unity — the largest term^{5.4} here is $(\partial u/\partial y)^2 \sim [1/\epsilon^2]$, so that we see $\nu/c_p \sim [\epsilon^2]$

We can then write the energy equation as

$$(5.14) \quad u \frac{\partial T}{\partial x} [1] + v \frac{\partial T}{\partial y} [\epsilon] [\epsilon^{-1}] = \alpha [\epsilon^2] \left(\frac{\partial^2 T}{\partial x^2} [1] + \frac{\partial^2 T}{\partial y^2} [\epsilon^{-2}] \right) + \frac{\nu}{c_p} [\epsilon^2] \Phi,$$

where

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 [1] + \left(\frac{\partial v}{\partial y} \right)^2 [1] \right] + \left(\frac{\partial v}{\partial x} [\epsilon] + \frac{\partial u}{\partial y} [\epsilon^{-1}] \right)^2.$$

Dropping terms of order $[\epsilon]$ and higher, this simplifies to

$$(5.15) \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{\nu}{c_p} \left(\frac{\partial u}{\partial y} \right)^2.$$

Eqs. (5.12), (5.13), and (5.15) are the so-called *boundary layer equations*.

^{5.4}The basic term $\partial u/\partial y$ is clearly of order $[1/\epsilon]$, but squaring this term raises it to $[1/\epsilon^2]$.

Reiterating some of the interesting characteristics of these equations:

- The pressure does not vary in the y direction.
- Since the pressure is constant in the y direction across the boundary layer, its value is equivalent to the value in the free-stream.
- Thus $P = P(x)$ and $\partial P/\partial x \rightarrow dP/dx$, which further implies that $P(x)$ can be determined from free-stream conditions so that dP/dx is a known quantity.
- No terms drop out of the continuity equation.

5.3. Dimensionless Parameters

Unfortunately, we are still faced with the fact that this system of equations is non-linear and cannot be solved with the mathematical techniques that we have utilized so far. There are advanced techniques that can be employed (e.g. similarity approaches that we outlined earlier), but usually a full-blown numerical approach is used. So why do we bother to study this problem here? Aside from developing an appreciation for the physical processes, our primary goal is to identify relevant non-dimensional parameters for the problem (much like we did for conduction) and to develop some important analogies between momentum and heat transfer. For the moment, we will neglect the viscous dissipation term.

As usual, let us define characteristic length, L , velocity, u_∞ , and temperature, T_∞ , scales for the problem at hand to non-dimensionalize the variables as

$$\begin{aligned} x^* &= \frac{x}{L}, & y^* &= \frac{y}{L}, \\ u^* &= \frac{u}{u_\infty}, & v^* &= \frac{v}{u_\infty}, \end{aligned}$$

and

$$T^* = \frac{T - T_s}{T_\infty - T_s}.$$

Performing the non-dimensionalization on the equations according to the results we developed in §4.4 on pp. 42, we find

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0,$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dP^*}{dx^*} + \frac{\nu}{u_\infty L} \frac{\partial^2 u^*}{\partial y^{*2}},$$

and

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\alpha}{u_\infty L} \frac{\partial^2 T^*}{\partial y^{*2}}.$$

From these equations, we can identify two non-dimensional parameters

- Reynolds number: defined as $Re = u_\infty L/\nu$
- Prandtl number: defined as $Pr = \nu/\alpha$

The Prandtl number is clear in light of the following from the energy equation

$$\frac{u_\infty L}{\alpha} = \frac{u_\infty L}{\alpha} \frac{\nu}{\nu} = \frac{u_\infty L}{\nu} \frac{\nu}{\alpha} = Re \cdot Pr.$$

Thus, the final non-dimensionalized form of the equations can be written

$$(5.16) \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0,$$

$$(5.17) \quad u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dP^*}{dx^*} + \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}},$$

and

$$(5.18) \quad u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re \cdot Pr} \frac{\partial^2 T^*}{\partial y^{*2}}.$$

The Reynolds number is familiar from fluid mechanics. It can be interpreted as the ratio of inertial forces to viscous forces, so it serves as a direct metric of how well frictional dissipation is able to dampen perturbations in a flow. This of course bears directly on whether a particular flow is laminar or turbulent.

Conversely, the Prandtl number is new — it appears only in the energy equation. Taking a closer look, we see that Pr is a direct property of the fluid itself. It does not contain any non-fluid properties, as for example the velocity and length scales in the Reynolds number. We can therefore speak of “the Prandtl number of a fluid”. How do we interpret this parameter physically? The Prandtl number is the ratio of kinematic viscosity ν to thermal diffusivity α , the former of which indicates the rate of momentum diffusion (from the viscous terms in the momentum equations) and the latter of which is the rate of thermal diffusion

$$Pr = \frac{\nu}{\alpha} = \frac{\text{momentum diffusion}}{\text{thermal diffusion}}.$$

We can say equivalently that Pr represents the ratio of the effectiveness of diffusional energy transport in the velocity boundary layer versus the thermal boundary layer. In laminar flows, we can thus infer

$$Pr \sim \frac{\delta}{\delta_t},$$

that is, Pr gives some approximate indication of the thickness of the velocity boundary layer versus the thermal boundary layer^{5.5} (Table 5.1).

Of course, we still have not lost anything in terms of non-linearity of the system in Eqs. (5.16) through (5.18), however, just as previously, the

^{5.5}This is not the case in turbulent flows where transport is a function of turbulent mixing, as well as diffusion.

TABLE 5.1. Approximate boundary layer relationships for various fluid types

Fluid	Pr	relationship
gases	~ 1	$\delta \approx \delta_t$
liquid metals	$\ll 1$	$\delta \ll \delta_t$
oils	$\gg 1$	$\delta \gg \delta_t$

number of relevant variables to any problem has been reduced. For example, the velocity distribution should be of the form

$$(5.19) \quad u^* = u^* \left(x^*, y^*, Re, \frac{dP^*}{dx^*} \right),$$

where dP^*/dx^* is considered to be determined by the geometry of the problem and the free-stream conditions and is considered a known parameter as mentioned previously.

What other dimensionless parameters arise for such configurations? Let us consider frictional drag on the body surface that arises because of viscous shear stress. Recall from fluid mechanics that the stress is defined as

$$\tau = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{\mu u_\infty}{L} \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0}.$$

This expressions allows us to derive the following form of the *skin friction factor*, defined by $C_f = \tau/(\rho u_\infty^2/2)$, as

$$C_f = \frac{2}{Re} \left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0}.$$

Eq. (5.19) indicates that $\partial u^*/\partial y^*$ evaluated at $y^* = 0$ is only a function of $(x^*, Re, dP^*/dx^*)$. Therefore, we arrive at the significant conclusion that the skin friction factor, an important engineering quantity, is only governed by three “universal” parameters according to the functional form

$$(5.20) \quad C_f = C_f \left(x^*, Re, \frac{dP^*}{dx^*} \right).$$

Likewise, we see that T^* is essentially a function of the form

$$(5.21) \quad T^* = T^* \left(x^*, y^*, Re, Pr, \frac{dP^*}{dx^*} \right),$$

so that it has the same dependencies of u^* , except with the addition of the Prandtl number as another parameter. Recall that we cast the convection coefficient in Eq. (5.3) on pp. 56 as

$$h = - \frac{k_f}{T_s - T_\infty} \left. \frac{\partial T}{\partial y} \right|_{y=0},$$

which, we can recast with dimensionless variables as

$$h = -\frac{k_f}{L} \frac{T_\infty - T_s}{T_s - T_\infty} \frac{\partial T^*}{\partial y^*} \Big|_{y^*=0} = \frac{k_f}{L} \frac{\partial T^*}{\partial y^*} \Big|_{y^*=0}.$$

This expression leads directly to the definition of the *Nusselt number*

$$Nu = \frac{\partial T^*}{\partial y^*} \Big|_{y^*=0} = \frac{hL}{k_f},$$

which is essentially the non-dimensional temperature gradient at the wall. As with the skin friction factor, y^* does not come into play at the wall, i.e. at $y^* = 0$, so that the Nusselt number depends only upon variables as

$$(5.22) \quad Nu = Nu \left(x^*, Re, Pr, \frac{dP^*}{dx^*} \right).$$

The Nusselt number, as we have written it here, depends on the local convection coefficient h , so that we refer to this form as the “local Nusselt number”. We can also write an average Nusselt number by simply substituting the average convection coefficient \bar{h} . Note that, while the form of the Nusselt number and the Biot number appear the same, k is that of the fluid for the Nusselt number (convection) and that of the solid for the Biot number (conduction).

IED Ex. 6.5
pp 351

5.4. Reynolds–Colburn Analogy

Now that we have introduced C_f , Nu , Re , and Pr as important dimensionless parameters in convection, we will develop an analogy among these parameters, so that we can get the answer to a convection problem by solving a fluid mechanics problem^{5.6}. Here, we will derive the *Reynolds–Colburn analogy* for laminar flow along a flat plate, which has the functional form

$$Nu = Nu(C_f, Re, Pr).$$

We will be able to calculate the Nusselt number from the Prandtl and Reynolds numbers and the skin friction factor. This is particularly useful, since measuring Re and C_f for the fluid mechanics problem is more straightforward than measuring Nu directly.

The rigorous development of this result relies on the exact solution of the boundary layer problem, as posed by Eqs. (5.16) through (5.18), where the the pressure gradient along the plate is assumed to vanish. We are not in a mathematical position to derive this solution, but we will rather use the approximate solution as a surrogate, so that the main concept can at least be illustrated^{5.7}. Assume, according to Figs. 5.1 and 5.2, that u_∞ is

^{5.6}Recall in Chapter 3 where the circuit analogy allowed us to get the answer to a heat conduction problem by solving an electrical circuit problem.

^{5.7}Most textbooks write down the exact solution directly in the form of the required dimensionless parameters (using the “please believe this” approach), so that the analogy is trivially obvious. We start with the approximate solution for velocity and temperature distribution. This is only a “please believe this temporarily” approach, as we will actually

free-stream velocity, y is the coordinate normal to the plate, δ is the velocity boundary layer thickness, and δ_t is the thermal boundary layer thickness. We start with an approximate solution for the boundary layer problem

$$(5.23) \quad u^* = \frac{u}{u_\infty} = \frac{3}{2} \frac{y}{\delta} - \frac{1}{2} \left(\frac{y}{\delta} \right)^3 \quad \text{where} \quad \delta = \sqrt{\frac{280}{13} \frac{\nu x}{u_\infty}} \approx \frac{4.64 x}{Re_x},$$

and

$$(5.24) \quad T^* = \frac{T - T_s}{T_\infty - T_s} = \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left(\frac{y}{\delta_t} \right)^3 \quad \text{where} \quad \delta_t = \frac{4.52 x}{Re_x^{1/2} Pr^{1/3}}.$$

Deriving the analogy is matter of manipulating the solutions appropriately. Writing the skin friction factor in the usual fashion as $C_f = \tau / (0.5\rho u_\infty^2)$, and substituting $\tau = \mu \partial u / \partial y = \nu \rho \partial u / \partial y$ at $y = 0$ from Newton's Law of Viscosity, we can write

$$C_f = \frac{2\nu}{u_\infty^2} \left. \frac{\partial u}{\partial y} \right|_{y=0}.$$

From the solution for the velocity profile in Eq. (5.23), we can solve for the velocity gradient at the wall as

$$\left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{3 u_\infty}{2 \delta},$$

so that a substitution yields C_f as

$$C_f = \frac{2\nu}{u_\infty^2} \frac{3 u_\infty}{2 \delta} = \frac{3\nu}{u_\infty \delta} = \frac{3\nu}{u_\infty} \sqrt{\frac{13}{280} \frac{u_\infty}{\nu x}} = \sqrt{\frac{117}{280} \frac{\nu}{u_\infty x}} = \frac{0.646}{Re_x^{1/2}}.$$

Again, the constant 0.646 represents the approximate solution and is very close to the value for the exact solution of 0.664, which we will now use. We therefore write $C_f = 0.664/Re_x^{1/2}$, or, in a more useful form

$$(5.25) \quad \frac{C_f}{2} = \frac{0.332}{Re_x^{1/2}}.$$

This is one-half of what is needed to complete the analogy.

Referring back to Eq. (5.3) on pp. 56, we insert the dimensionless temperature into the gradient to obtain

$$h = - \frac{k_f}{T_s - T_\infty} \left. \frac{\partial T}{\partial y} \right|_{y=0} = k_f \left. \frac{\partial T^*}{\partial y} \right|_{y=0},$$

From the solution of the temperature profile in Eq. (5.24), we can solve for the temperature gradient at the wall as

$$\left. \frac{\partial T^*}{\partial y} \right|_{y=0} = \frac{3}{2 \delta_t},$$

derive this solution in the following chapter. The only non-rigorous component of the process is where we substitute a certain constant derived from the exact solution for its counterpart from the approximate solution. This will be pointed out.

so that substitution yields h as

$$h = \frac{3 k_f}{2 \delta_t}.$$

Plugging in δ_t , we find

$$h \approx \frac{0.332 k_f Re_x^{1/2} Pr^{1/3}}{x},$$

from which we can manipulate to write in terms of the Nusselt number as

$$(5.26) \quad Nu = \frac{h x}{k_f} = 0.332 Re_x^{1/2} Pr^{1/3}.$$

Combining Eqs. (5.25) and (5.26), we can write

$$(5.27) \quad Nu = \frac{C_f Re Pr^{1/3}}{2},$$

which is the Reynolds–Colburn analogy. This equation is applicable for laminar flat plate flow in the range of approximately $0.6 < Pr < 60$ (Incropera and Dewitt, 2002).

CHAPTER 6

External Convection

Now we would like to focus on convection problems of external flow more closely. This includes cases such as flow over a flat plate, airfoil, etc. We will concentrate on forced convection, meaning that a pressure gradient or some other externally-applied force powers the flow (as opposed to buoyancy effects). Once again, the primary objective is to determine the convection coefficient, which may be found via the Nusselt number $Nu_x = Nu_x(x^*, Re_x, Pr)$. Of course, there are a number of approaches for doing this

- theoretical: apply analytical techniques to solve governing equations (this is the approach we exploited for all of our conduction problems)
- empirical: perform experimental measurements under controlled conditions and correlate data in terms of relevant dimensionless parameters
- numerical: approximate governing equations with discrete expressions and use computers to solve iteratively

As we have discussed in Chapter 5, convection problems are usually non-linear, so the success of theoretical approaches will be limited to a few specific cases. Conversely, empirical techniques lend themselves well to these sorts of problems. We have already determined the relevant dimensionless groups for the typical convection problem. Given sufficient measurement data, results can be cast in the form of *correlations*, for example, equations of the form

$$\overline{Nu_x} = C Re_L^m Pr^n ,$$

where $\overline{Nu_x}$ is the averaged Nusselt number, and C , m , and n are problem-specific constants that must be determined.

However, this sort of approach presents an immediate and obvious problem. It is well-known that fluid properties, for example viscosity and conductivity, vary according to temperature. How do we properly account for such variations in a correlation equation? One method is to evaluate properties at the so-called *film temperature*

$$(6.1) \quad T_f = \frac{T_s + T_\infty}{2} ,$$

which is simply the average of the temperature extremes. Another possibility is the incorporation of additional correlation terms, for example of the form $(\mu_\infty/\mu_s)^r$, to account for such variations.

6.1. Laminar Flow Over A Flat Plate

Before delving further into correlations, we will discuss cases that can be treated purely via theory. Flow over a flat plate is one such configuration (Figs. 5.1 and 5.2). Assuming no pressure gradient along the plate and no viscous dissipation, the boundary layer form of the equations given by Eqs. (5.12), (5.13), and (5.15) on pp. 59 can be written as

$$(6.2) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2},$$

$$(6.3) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

and

$$(6.4) \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}.$$

In Chapter 5, we derived the Reynolds–Colburn analogy using analytical solutions, both approximate and exact, to this problem. Where do these solutions come from? The exact solution is not trivial, in fact it is beyond the mathematical capabilities we have developed thus far. Again, this is a result of the non–linearity of the system in Eqs. (5.16) through (5.18) on pp. 61. We can cast the solution procedure, but will be unable to follow it to completion.

The similarity technique introduced in §4.6 can be used to reduce the partial differential system to an ordinary differential equation. As we mentioned previously, one should suspect that a similarity approach may work when no inherent scales can be identified in the problem. In this case, the transformation variables

$$\eta = y \sqrt{\frac{u_\infty}{\nu x}} \quad \text{and}$$

$$f(\eta) = \frac{\psi}{u_\infty} \sqrt{\frac{\nu x}{u_\infty}}$$

are appropriate, where $\psi = \psi(x, y)$ is the *stream function*, which is defined as

$$(6.5) \quad u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}.$$

Transforming the momentum equation appropriately, we find it becomes

$$2f''' + ff'' = 0,$$

where the prime symbol indicates the regular derivative of a uni-variate function. Unfortunately, unlike the conduction problem in §4.6, this ordinary differential equation is still non-linear. This equation can be solved in a number of ways^{6.1} to yield the solution given in Table 6.1. The resulting

TABLE 6.1. Boundary layer solution

η	f	$f' = u^*$	f''
0	0	0	0.332
0.4	0.027	0.133	0.331
0.8	0.106	0.265	0.327
1.2	0.238	0.394	0.317
1.6	0.420	0.517	0.297
2.0	0.650	0.630	0.267
2.4	0.922	0.729	0.228
2.8	1.231	0.812	0.184
3.2	1.569	0.876	0.139
3.6	1.930	0.923	0.098
4.0	2.306	0.956	0.064
4.4	2.692	0.976	0.039
4.8	3.085	0.988	0.022
5.2	3.482	0.994	0.011
5.6	3.880	0.997	0.005
6.0	4.280	0.999	0.002
6.4	4.679	1.000	0.001
6.8	5.079	1.000	0.000

boundary layer thickness is

$$(6.6) \quad \delta = \frac{5x}{\sqrt{Re_x}}$$

and the resulting skin friction coefficient is

$$(6.7) \quad C_f = \frac{0.664}{\sqrt{Re_x}}.$$

Conversely, if we take a closer look at the energy equation in Eq. (5.18) on pp. 61, we will see that it is linear, because the velocity components are known. Using the similarity variables, we can transform this equation into

$$T^{*''} + \frac{Pr}{2} f T^{*' } = 0,$$

where $T^* = T^*(\eta)$ and the prime symbol again denotes the regular derivative of a uni-variate function. This can be solved, for example by numerical

^{6.1}Numerically, this equation is fairly straightforward to solve, e.g. with the so-called “shooting method”. Conversely, a series expansion technique can also be used to solve to any desired degree of precision.

integration for a specific Prandtl number, to yield T^* . The dimensionless results can be cast as

$$(6.8) \quad Nu_x = \frac{h_x x}{k_f} = 0.332 Re_x^{1/2} Pr^{1/3} \quad \text{for} \quad Pr \geq 0.6 .$$

A number of additional results are shown in Incropera and Dewitt (2002), in particular the local Nusselt number Nu_x in Eq. (6.8) can be integrated to obtain an averaged value of

$$(6.9) \quad \overline{Nu_x} = 2 Nu_x .$$

6.2. Karman–Pohlhausen Integral Solution

The fact that the exact solution in the previous section is difficult from a mathematical standpoint suggests that an approximate solution might be useful. T. von Karman (1921) and K. Pohlhausen (1921) derived just such a solution in the form of an integral method that can be solved to various degrees of accuracy. In fact, we have already used this solution in the form of Eqs. (5.23) and (5.24) on pp. 64 for the Reynolds–Colburn analogy. First, we start with the continuity equation

$$(6.10) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and integrate it across the boundary layer:

$$(6.11) \quad \int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta \frac{\partial v}{\partial y} dy = 0 .$$

Now note that $v = 0$ at $y = 0$ (as part of the no-slip boundary condition) and that the second part of the equation is simply dv because of the chain rule. This gives:

$$(6.12) \quad v|_{y=\delta} = - \int_0^\delta \frac{\partial u}{\partial x} dy .$$

Now, we also integrate the boundary layer form of the momentum equation (6.2) over the boundary layer

$$(6.13) \quad \int_0^\delta u \frac{\partial u}{\partial x} dy + \int_0^\delta v \frac{\partial u}{\partial y} dy = \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2} dy ,$$

which gives, after integrating the second term on the left hand side by parts^{6.2} and recognizing the right hand side can be integrated directly,

$$(6.14) \quad \int_0^\delta u \frac{\partial u}{\partial x} dy + uv|_{y=0}^{y=\delta} - \int_0^\delta u \frac{\partial v}{\partial y} dy = \nu \frac{\partial u}{\partial y} \Big|_{y=0}^{y=\delta} .$$

^{6.2}If integrating by parts, we can consider $\frac{\partial u}{\partial y} dy$ as simply du by the Chain Rule, which means we are evaluating $\int v du$. Applying integration by parts, we see $\int v du = uv - \int u dv$. However, $\frac{\partial v}{\partial y} dy = dv$, again by the Chain Rule. This gives the second and third terms on the left hand side of Eq. (6.14).

The first term remains the same, however, to further develop Eq. (6.14), we make the following observations. The second term can be evaluated using Eq. (6.12) for v and observing that $u = 0$ at $y = 0$ and $u = u_0$ at $y = \delta$. In the third term, we utilize the original continuity equation (6.10) to swap $\partial v/\partial y = -\partial u/\partial x$. The fourth term is evaluated using the fact that $\partial u/\partial y$ is zero at $y = \delta$. These modifications give

$$(6.15) \quad \int_0^\delta u \frac{\partial u}{\partial x} dy - u_0 \int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta u \frac{\partial u}{\partial x} dy = -\nu \frac{\partial u}{\partial y} \Big|_{y=0},$$

which can be simplified to

$$(6.16) \quad u_0 \int_0^\delta \frac{\partial u}{\partial x} dy - \int_0^\delta 2u \frac{\partial u}{\partial x} dy = \nu \frac{\partial u}{\partial y} \Big|_{y=0}.$$

The u_0 can be taken directly under the integral and the u can be “integrated” in^{6.3}, which yields:

$$(6.17) \quad \int_0^\delta \frac{\partial}{\partial x} (u_0 \cdot u - u \cdot u) dy = \nu \frac{\partial u}{\partial y} \Big|_{y=0}.$$

Now, the right hand side is basically a term that is evaluated, and the differential on the left hand side can be taken outside the integral (since the term is an integral w.r.t. y) and done later, therefore $\partial/\partial x \rightarrow d/dx$, which yields:

$$(6.18) \quad \frac{d}{dx} \int_0^\delta (u_0 - u) \cdot u dy = \nu \frac{\partial u}{\partial y} \Big|_{y=0}.$$

Eq. (6.18) is an integral form of the boundary layer equations. If we examine this equation carefully, we see that the right hand side is basically the shear stress at the surface of the plate. That is,

$$(6.19) \quad \frac{d}{dx} \int_0^\delta (u_0 - u) \cdot u dy = \tau_w.$$

This quantity is unknown because we do not know the velocity profile u . Similarly, the left hand side also cannot be evaluated because it contains u . We apparently have an implicit equation in u that cannot be solved in closed form.

However, we can apply reasonable approximations for u . For example, let us assume that u is described by a polynomial^{6.4}. We know that the flow is developing, therefore it is a function of both x and y , i.e. $u = u(x, y)$. However, we would like to use a polynomial that is a function of just a single

^{6.3}i.e. $2u \frac{\partial u}{\partial x} = \frac{\partial u^2}{\partial x}$

^{6.4}Here is where the idea of various degrees of accuracy enters into the problem. Specifically, lower-order polynomials will, in general, yield less accurate results as compared to higher-order polynomials. The order is limited by how many boundary conditions we can identify because these must be used to evaluate the polynomial coefficients.

variable. Let us then assume the following: choose a third-order polynomial having 4 constants^{6.5}

$$(6.20) \quad \frac{u}{u_0} = a_1 + a_2 \left(\frac{y}{\delta}\right) + a_3 \left(\frac{y}{\delta}\right)^2 + a_4 \left(\frac{y}{\delta}\right)^3 .$$

Eq. (6.20) appears to depend only upon y , however, since the boundary layer thickness depends upon x , i.e. $\delta = \delta(x)$ as shown in Eq. (6.6), our solution for u/u_0 is a function of both x and y . Let us describe the boundary conditions that we will use to evaluate the coefficients in Eq. (6.20):

- No-slip boundary conditions at the wall give

$$(6.21) \quad u|_{y=0} = 0 .$$

- The gradient of velocity vanishes at the edge of the boundary layer. In other words, the rate of change of u with respect to y goes to zero at $y \geq \delta$. This gives

$$(6.22) \quad \left. \frac{\partial u}{\partial y} \right|_{y=\delta} = 0 .$$

- The velocity at the edge of the boundary layer is known to be u_0 , which gives

$$(6.23) \quad u|_{y=\delta} = u_0 .$$

- While these 3 boundary conditions are fairly obvious, a fourth can be deduced with a little effort. Looking back to the boundary layer momentum equation (6.2) we see that this can be *evaluated* at $y = 0$ using, once again, the concept that there is no-slip at the plate surface. Therefore, plugging in $u = v = 0$ at $y = 0$, we find

$$(6.24) \quad \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0} = 0 .$$

Now, we have four unknowns: $a_1 \dots a_4$, and four equations with which to evaluate them: (6.21) through (6.24). Carrying this out, we find $a_1 = a_3 = 0$, $a_2 = 1.5$, and $a_4 = -0.5$, which gives

$$(6.25) \quad \frac{u}{u_0} = \frac{3}{2} \left(\frac{y}{\delta}\right) - \frac{1}{2} \left(\frac{y}{\delta}\right)^3 .$$

Note that this equation is the first part of the solution shown back in Eq. (5.23). We now know the *form* of u/u_0 , but since we do not know how δ varies, we still do not know the explicit solution for u/u_0 ! This is where we will now apply the integral form of the boundary layer equations we derived: i.e. Eq. (6.18).

We use Eq. (6.25) to now evaluate the terms in Eq. (6.18). Let us show this explicitly:

^{6.5}There are three boundary conditions that can be easily identified, and a fourth that can be deduced without too much difficulty, as we shall see shortly.

- From Eq. (6.25), we write u as

$$u = u_0 \left(\frac{1.5}{\delta} y - \frac{0.5}{\delta^3} y^3 \right)$$

- Starting inside the integral on the left hand side we can then write

$$u_0 - u = u_0 \left(1 - \frac{1.5}{\delta} y + \frac{0.5}{\delta^3} y^3 \right)$$

so that

$$(u_0 - u) \cdot u = u_0^2 \left(1 - \frac{1.5}{\delta} y + \frac{0.5}{\delta^3} y^3 \right) \left(\frac{1.5}{\delta} y - \frac{0.5}{\delta^3} y^3 \right)$$

- We can then evaluate the integral on the left hand side of Eq. (6.18) as

$$\int_0^\delta (u_0 - u) \cdot u \, dy = \int_0^\delta u_0^2 \left(\frac{1.5}{\delta} y - \frac{2.25}{\delta^2} y^2 - \frac{0.5}{\delta^3} y^3 + \frac{1.5}{\delta^4} y^4 - \frac{0.25}{\delta^6} y^6 \right) dy$$

which is

$$u_0^2 \left(\frac{3}{4\delta} y^2 - \frac{3}{4\delta^2} y^3 - \frac{1}{8\delta^3} y^4 + \frac{3}{10\delta^4} y^5 - \frac{1}{28\delta^6} y^7 \right) \Big|_{y=0}^\delta$$

which simplifies to

$$\frac{39}{280} u_0^2 \delta$$

- The right hand side of Eq. (6.18) is easily evaluated as

$$\frac{3}{2} \frac{\nu u_0}{\delta}$$

- Eq. (6.18) can then be written as

$$\frac{d}{dx} \left(\frac{39}{280} u_0^2 \delta \right) = \frac{3}{2} \frac{\nu u_0}{\delta}$$

- We note that δ is the only term on the left hand side that is a function of x . Everything else is constant with respect to x , therefore we can write the equation as

$$\left(\frac{39}{280} u_0^2 \right) \frac{d\delta}{dx} = \frac{3}{2} \frac{\nu u_0}{\delta}$$

which is a simple separable differential equation that can be written as

$$\delta \, d\delta = \frac{140}{13} \frac{\nu}{u_0} dx$$

- This equation can be integrated along the flow direction, that is, in x as

$$\int \delta \, d\delta = \int \frac{140}{13} \frac{\nu}{u_0} dx$$

from which we find

$$\frac{\delta^2}{2} = \frac{140}{13} \frac{\nu x}{u_0} + C_0$$

where C_0 is a constant.

- Recalling the fact that the boundary layer thickness is zero at the leading edge of the plate, that is $\delta(0) = 0$, we find $C_0 = 0$ so that

$$\frac{\delta^2}{2} = \frac{140}{13} \frac{\nu x}{u_0}$$

which can be written as

$$(6.26) \quad \delta = \sqrt{\frac{280}{13} \frac{\nu x}{u_0}} = \sqrt{\frac{280}{13} \frac{\nu x}{u_0 x^2} x^2} = 4.640955 \sqrt{\frac{\nu}{u_0} x}$$

Note that this equation is the second and concluding part of the solution shown back in Eq. (5.23). We see that the second part of Eq. (6.26) is simply the inverse of the square root of the Reynolds number based on x . We can therefore write the boundary layer solution as

$$(6.27) \quad \delta \approx \frac{4.64 x}{Re^{1/2}}$$

We have now solved the problem, since Eq. (6.27), along with Eq. (6.25) describe the complete velocity profile. Note that the constant 4.64 is remarkably close to the value of 5.0 that was obtained from the full non-linear (and much more difficult) exact solution! Using Eq. (6.19), we can also evaluate the shear stress and the wall skin friction coefficient C_f , for example

$$(6.28) \quad C_f = \frac{\tau_w}{\rho u_0^2/2} = \frac{0.646}{\sqrt{Re_x}},$$

which is, once again, quite comparable to the exact solution in Eq. (6.7).

We assumed a third-order polynomial for the velocity profile in Eq. (6.20). Of course, other forms could have been assumed as well. Table 6.2 compares the exact solution to several approximate ones, including the cubic polynomial we have used here. The procedure to solve Eq. (6.4) for temperature is

TABLE 6.2. Laminar boundary layer solutions

solution	$\delta\sqrt{Re_x}/x$	$C_f\sqrt{Re_x}$
exact	5.0	0.664
linear	3.46	0.578
parabolic	5.48	0.730
cubic	4.64	0.646
sinusoidal	4.79	0.655

somewhat similar, but more tedious and leads to Eq. (5.24). Development of this result is shown in a number of reference texts (e.g. Burmeister, 1983, §8.3).

6.3. Empirical Correlations

Now that we have comprehensively studied laminar flow over a flat plate, we are compelled to admit our inability to examine more complicated configurations from a purely theoretical perspective. However, we have characterized boundary layer-type problems in terms of the relevant dimensionless parameters and this provides a basis for correlating experimental measurements for various configurations. We do not delve into the details of making such measurements. Rather, we simply catalog some of the notable formulae for basic problems. Various texts have more extensive listings (e.g. Özişik, 1985; Incropera and Dewitt, 2002). These texts also list the original references for the correlations presented below.

It is important to understand the limitations of these formulae. *They are not general solutions!* They are simply relationships that have been found to be reasonably accurate in some limited range of the parameters^{6.6}. Therefore, when solving problems, it is always important to make sure its parameters correspond to the correlation that is chosen. Unless otherwise noted, these results are based on evaluating fluid properties at the film temperature.

6.3.1. Flat Plate. Theoretical results for the laminar flat plate were restricted to $Pr > 0.6$, however, recall from Table 5.1 that liquid metals can have substantially lower Prandtl numbers. In this case, the thermal boundary layer develops very rapidly, so that $\delta_t \gg \delta$. We can assume then that $u = u_\infty$ over the whole thermal boundary layer and rework the thermal boundary layer solution. This yields

$$(6.29) \quad Nu_x = 0.565 Pe_x^{1/2} \quad Pr \leq 0.05 \quad \text{and} \quad Pe_x \geq 100,$$

where $Pe_x = Re_x Pr$ is the *Peclet number*. A correlation valid for laminar flow and all Pr is given by

$$(6.30) \quad Nu_x = \frac{0.3387 Re_x^{1/2} Pr^{1/3}}{\left(1 + [0.0468/Pr]^{2/3}\right)^{1/4}} \quad Pe_x \geq 100.$$

For turbulent flow, the correlation

$$(6.31) \quad Nu_x = 0.0296 Re_x^{4/5} Pr^{1/3} \quad 0.6 \leq Pr \leq 60 \quad \text{and} \quad Re_x \leq 10^7$$

is reasonably accurate.

Eqs. (6.29) through (6.31) are valid for the local Nusselt number. Correlations for the averaged Nusselt number are somewhat more difficult because they have to account for the fact that a boundary layer initially has a laminar behavior, undergoes transition, and then becomes turbulent^{6.7}. One

^{6.6}Accuracy may be on the order of about 20 %.

^{6.7}Of course, if the area of interest is strictly laminar or turbulent, the appropriate averaged Nusselt number can be obtained by direct integration according to Eq. (5.2) on pp. 54.

way to handle this is to assume that transition occurs at a specific critical location along the plate x_c , which divides the plate into laminar and turbulent regions. According to Eq. (5.2) on pp. 54, we could then model the overall convection coefficient as

$$(6.32) \quad \bar{h} = \frac{1}{L} \left(\int_0^{x_c} h_{lam} dx + \int_{x_c}^L h_{turb} dx \right).$$

Using h_{lam} based on Eq. (6.8) and h_{turb} based on Eq. (6.31), we find

$$(6.33) \quad \overline{Nu}_L = \left(0.037 Re_L^{4/5} - 871 \right) Pr^{1/3},$$

I&D Ex. 7.1
pp 400

where the limits are $0.6 \leq Pr \leq 60$ and the end of the plate at $x = L$ is assumed to lie in the turbulent zone, i.e. $5 \times 10^5 \leq Re \leq 10^8$. (We assume that transition occurs at $Re = 5 \times 10^5$.)

6.3.2. Cylinders in Cross-Flow. Engineering applications frequently involve cross-flow over one or more cylinders (Fig. 6.1). As you might recall

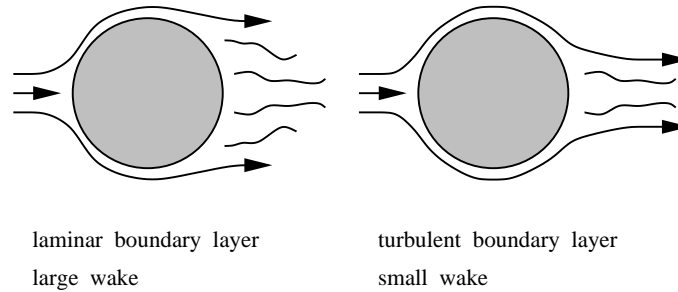


FIGURE 6.1. *Laminar and turbulent boundary layer formation for cross-flow over a cylinder.*

from fluid mechanics, flow separation occurs at some point where the boundary layer detaches from the surface, forming a low-pressure wake. The flow is characterized by the Reynolds number

$$Re_D = \frac{u_\infty D}{\nu},$$

where u_∞ is the velocity upstream of the cylinder and D is the diameter, and by the coefficient of drag

$$C_D = \frac{F_D}{A_f \rho u_\infty^2 / 2},$$

where F_D is the measured drag force and A_f is the projected frontal area normal to the flow. The character of the boundary layer (laminar or turbulent) strongly influences where separation occurs, and thus the magnitude of C_D (Fig. 6.2). There is a significant decrease as transition and turbulent flow become important at Reynolds numbers above 200,000 to 500,000. These

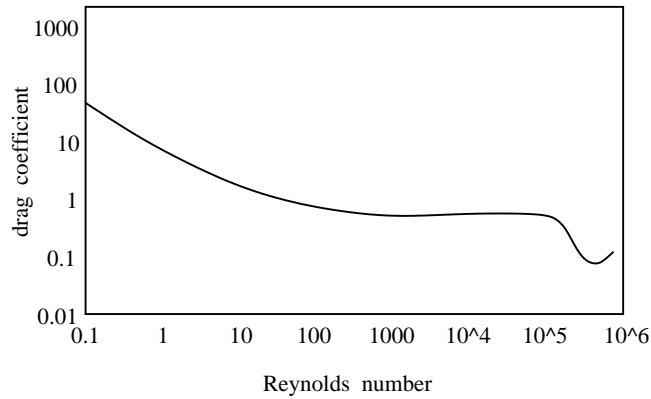


FIGURE 6.2. *Coefficient of drag for cross-flow over a circular cylinder.*

features strongly influence the convection characteristics (e.g. Incropera and Dewitt, 2002, Fig. 7.9, pp. 410).

An overall Nusselt number, valid over all Reynolds numbers Re_D for which data are available, has been proposed as

$$(6.34) \quad \overline{Nu_D} = 0.3 + \frac{0.62 Re_D^{1/2} Pr^{1/3}}{\left(1 + [0.4/Pr]^{2/3}\right)^{1/4}} \left[1 + \left(\frac{Re_D}{282,000}\right)^{5/8}\right]^{4/5}.$$

The restriction on the Prandtl number is roughly $Re_D Pr > 0.2$. Again, fluid properties are evaluated at the film temperature. Incropera and Dewitt (2002) list several additional correlation equations that are appropriate for groups of more than a single cylinder.

6.3.3. Spheres. Boundary layer effects for flow over spheres are similar to those described for cylinders. The primary correlation for averaged Nusselt number is

$$(6.35) \quad \overline{Nu_D} = 2 + \left(0.4 Re_D^{1/2} + 0.06 Re_D^{2/3}\right) Pr^{0.4} \left(\frac{\mu}{\mu_s}\right)^{1/4},$$

where all properties are evaluated at T_∞ , except μ_s , which is evaluated at the sphere surface temperature. The restrictions on Eq. (6.35) are approximately $0.71 < Pr < 380$ and $3.5 < Re_D < 7.6 \times 10^4$ (Incropera and Dewitt, 2002).

I&D Ex. 7.5
pp 415

CHAPTER 7

Internal Convection

In Chapter 6, we examined convection in external flows, where the growth of the boundary layer is not constrained. Here, we shall study configurations of internal flow in which the boundary layer is not permitted to grow without limitation, e.g. pipes, ducts, etc. Now, in addition to the consideration of laminar versus turbulent flow, we must be concerned with whether the flow is developing or fully developed^{7.1}. We will find that, unlike for external flows, certain internal flow configurations can be solved exactly in closed form since non-linear terms in the momentum equations may disappear. These cases, as you probably remember from fluid mechanics, are always those in which the flow is fully developed and thus the velocity vector is only a function of the streamwise velocity component. For configurations where the flow remains in a developing state, solution procedures will be somewhat limited and we shall be compelled to rely more on empirical data and correlations.

7.1. Laminar Pipe Flow

Let us start with the example of fully developed laminar flow in a circular pipe (Fig. 7.1). As fluid enters the pipe, a boundary layer begins to form

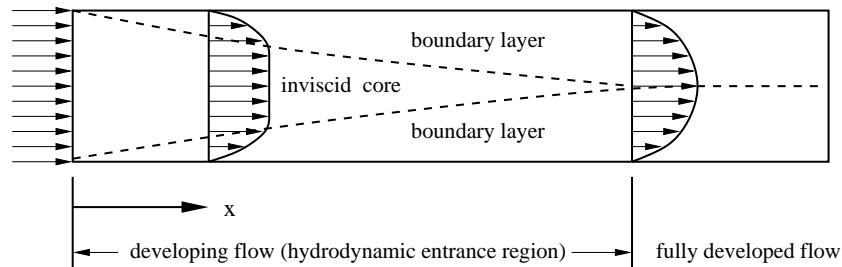


FIGURE 7.1. *Flow in a pipe with developing and fully developed regions.*

and grow inward from the periphery, eventually merging so as to encompass the entire cross-sectional area of the pipe. The distance x_{fd} at which this

^{7.1}Recall from fluid mechanics that flow along the x coordinate direction is developing if $\partial/\partial x \neq 0$, whereas it is fully developed if $\partial/\partial x = 0$ (with the exception of the pressure gradient, which need not vanish).

occurs is the hydrodynamic entry length. For $x \geq x_{fd}$, the flow is fully developed, meaning quantities^{7.2} no longer change as a function of x , and viscous effects are non-negligible throughout the entire cross-section of the pipe.

The Reynolds number characterizing pipe flows is based on the pipe diameter D and the average, or mean velocity over the cross-section \bar{u}

$$Re_D = \frac{\bar{u} D}{\nu} .$$

Transition from laminar to turbulent flow is usually in the neighborhood of $Re_D \approx 2300$. For laminar flow, x_{fd} is on the order

$$x_{fd} \approx 0.05 Re_D D ,$$

while for turbulent flow it is in the approximate range

$$10 D < x_{fd} < 60 D .$$

We might recall from fluid mechanics that for fully developed laminar flow, the velocity vector simplifies to $u = u(r)$. That is, the radial and azimuthal velocity components vanish, leaving only the streamwise component u . Consequently, the governing equation for u is obtained by simplifying the streamwise momentum equation

$$(7.1) \quad \frac{\mu}{r} \frac{d}{dr} \left(r \frac{du}{dr} \right) = \frac{dP}{dx} .$$

This is a second-order ordinary differential equation — recall that the flow is driven by the pressure gradient dP/dx , which is taken as a known quantity. Therefore, we simply integrate twice to obtain the velocity distribution

$$(7.2) \quad u(r) = \frac{1}{\mu} \left(\frac{dP}{dx} \right) \frac{r^2}{4} + C_1 \ln r + C_2 ,$$

where C_1 and C_2 are constants of integration. As usual, we must use boundary conditions to evaluate them. One boundary condition is immediately available from the no-slip requirement, $u = 0$ at $r = r_0$, where r_0 is the pipe radius. However, this is a situation in which the second boundary condition is not completely obvious. It can be derived from either of the following physical observations:

- the flow is well-behaved, so that the natural log term must remain finite at $r = 0$
- because of symmetry of the overall problem, the velocity profile must also be symmetric about $r = 0$

Both these observations yield the same result that $C_1 = 0$. For example,

$$u|_{r=0} = 0 + C_1 \ln 0 + C_2 = \text{finite} \quad C_1 = 0 .$$

^{7.2}Again, except for the pressure gradient, which cannot vanish, otherwise there would be no driving force to sustain the flow.

Applying no-slip, we find

$$u|_{r=r_0} = \frac{1}{\mu} \left(\frac{dP}{dx} \right) \frac{r_0^2}{4} + C_2 = 0,$$

so that

$$C_2 = -\frac{1}{\mu} \left(\frac{dP}{dx} \right) \frac{r_0^2}{4}$$

Substituting C_1 and C_2 into the general solution given by Eq. (7.2), we find the exact solution is

$$u(r) = \frac{1}{\mu} \left(\frac{dP}{dx} \right) \frac{r^2}{4} - \frac{1}{\mu} \left(\frac{dP}{dx} \right) \frac{r_0^2}{4},$$

or with further simplification

$$(7.3) \quad u(r) = -\frac{1}{4\mu} \left(\frac{dP}{dx} \right) r_0^2 \left[1 - \left(\frac{r}{r_0} \right)^2 \right].$$

Note that the pressure gradient must be negative to drive the flow in the positive x direction, i.e. the flow moves from high pressure to low pressure. Volume flow rate can be calculated by integrating the velocity profile as

$$(7.4) \quad Q = -\frac{\pi r_0^4}{8\mu} \frac{dP}{dx}$$

as shown in Appendix B. Again, the negative sign in Eq. (7.4) is compensated for by the negative pressure gradient dP/dx . According to the definition $Q = \bar{u}A$, we immediately have the average, or mean velocity as

$$(7.5) \quad \bar{u} = \frac{Q}{A} = \frac{Q}{\pi r_0^2} = -\frac{r_0^2}{8\mu} \frac{dP}{dx} = -\frac{D^2}{32\mu} \frac{dP}{dx}.$$

We have rewritten the result in terms of the pipe diameter D in the last expression. From this result, we see that the exact solution in Eq. (7.3) can be expressed in the equivalent form

$$(7.6) \quad u(r) = 2\bar{u} \left[1 - \left(\frac{r}{r_0} \right)^2 \right].$$

Also, the mass flow rate $\dot{m} = \rho\bar{u}A = \rho Q$ is

$$(7.7) \quad \dot{m} = -\frac{\rho\pi r_0^4}{8\mu} \frac{dP}{dx}$$

Furthermore, Eqs. (7.3) and (7.6) indicate that the maximum velocity, which occurs at $r = 0$, is

$$(7.8) \quad u_{max} = u(0) = -\frac{r_0^2}{4\mu} \frac{dP}{dx} = 2\bar{u}.$$

Now that the fluid mechanics problem is solved, we can turn our attention to the heat transfer problem. The immediate trouble we see is that, unlike for external flows, there's no fixed free-stream reference temperature

T_∞ . The convention is instead to use a mean or “bulk” temperature defined as

$$\int_{A_{cs}} \rho u c_v T dA_{cs} = \dot{m} c_v T_m .$$

From this expression, we see that T_m is based on the thermal energy transported as the flow moves through a cross-section of the pipe^{7.3}. Let us assume constant values for density ρ and specific heat c_v . Moreover, $dA_{cs} = 2\pi r dr$ and $A_{cs} = \pi r_0^2$, which gives

$$T_m = \frac{\rho c_v \int_0^{r_0} u T 2\pi r dr}{\rho \bar{u} \pi r_0^2 c_v} ,$$

or simplifying

$$(7.9) \quad T_m = \frac{2}{\bar{u} r_0^2} \int_0^{r_0} u T r dr .$$

This is now a suitable quantity to use in our definition of the convective heat transfer coefficient, which we define for this problem as (Özişik, 1985, pp. 243)

$$(7.10) \quad q_s'' = h(T_m - T_s) ,$$

where T_s and q_s'' are the temperature and heat flux at the surface of the pipe, respectively.

7.2. The Case of Constant Heat Flux

Note that unlike T_∞ for external flows, T_m as defined by Eq. (7.9) is generally not constant, but varies with axial distance. If the fluid is being heated ($T_s > T_m$), then T_m will increase along the flow direction, and vice versa. If that is the case, then clearly $dT_m/dx \neq 0$, so it appears that we can never realize the fully developed condition for the convection problem. Fortunately, this apparent contradiction is resolved if we work with dimensionless temperature. Let us define

$$(7.11) \quad \theta = \frac{T(r, x) - T_s(x)}{T_m(x) - T_s(x)} .$$

We then take fully developed thermal conditions to be defined by $\partial\theta/\partial x = 0$, which can be obtained for the case of constant wall heat flux q_s'' .

Let us now deduce some consequences laid out by Eq. (7.11). If $\partial\theta/\partial x = 0$, then θ cannot be a function of x for thermally fully developed flow. It follows that $\partial\theta/\partial r$ cannot likewise be a function of x . We can evaluate

$$\left. \frac{\partial\theta}{\partial r} \right|_{r=r_0} = \left. \frac{\partial}{\partial r} \left[\frac{T(r, x) - T_s(x)}{T_m(x) - T_s(x)} \right] \right|_{r=r_0} = \frac{1}{T_m(x) - T_s(x)} \left. \frac{\partial T}{\partial r} \right|_{r=r_0} ,$$

^{7.3}That is, ρu is the mass flux and $c_v T$ is the energy per unit mass. Integrating their product, the energy flux, over the cross-section gives the rate at which energy is transported through the cross-section.

and by the same argument conclude that this result is not a function of x . In other words, this expression remains fixed as the flow proceeds in the x direction. If we substitute surface heat flux from Newton's Law of Cooling in Eq. (7.10), $q_s''/h = (T_m - T_s)$, and from Fourier's Law of Conduction, $\partial T/\partial r = -q_s''/k$ at $r = r_0$ then we find

$$\frac{1}{T_m(x) - T_s(x)} \frac{\partial T}{\partial r} \Big|_{r=r_0} = \frac{h}{q_s''} \left[-\frac{q_s''}{k} \Big|_{r=r_0} \right] = \left| \frac{h}{k} \right|,$$

from which we must conclude that the local convection coefficient h is independent of x for fully developed flow^{7.4}.

Furthermore, Eq. (7.10) indicates that for constant h and constant q_s'' , the difference $T_s - T_m$ must be a constant that does not vary with x . We already know that T_s and T_m change with x , so that, if their difference is a constant, the rates of change of these two temperatures must be the same in the fully developed regime, i.e.

$$(7.12) \quad \frac{dT_m}{dx} = \frac{dT_s}{dx}.$$

Let us go one step further. We evaluate $\partial\theta/\partial x$ using θ as defined in Eq. (7.11). Our initial assumption of fully developed conditions requires $\partial\theta/\partial x = 0$. Evaluating, we find

$$\begin{aligned} \frac{\partial\theta}{\partial x} &= \frac{\partial}{\partial x} \left[\frac{T(r, x) - T_s(x)}{T_m(x) - T_s(x)} \right] \\ &= \frac{\partial}{\partial x} \left([T(r, x) - T_s(x)] [T_m(x) - T_s(x)]^{-1} \right) \\ &= \frac{1}{T_m(x) - T_s(x)} \frac{\partial [T(r, x) - T_s(x)]}{\partial x} \\ &\quad - \frac{T(r, x) - T_s(x)}{[T_m(x) - T_s(x)]^2} \frac{\partial [T_m(x) - T_s(x)]}{\partial x} \\ &= 0, \end{aligned}$$

The second term vanishes since $\partial/\partial x$ of $[T_m(x) - T_s(x)]$ was just shown to be 0 in Eq. (7.12). This leaves

$$\frac{1}{T_m(x) - T_s(x)} \frac{\partial [T(r, x) - T_s(x)]}{\partial x} = 0,$$

where $T_m(x) - T_s(x)$ is, again, a non-zero constant according to Eq. (7.10). We can therefore write more concisely

$$\frac{\partial [T(r, x) - T_s(x)]}{\partial x} = 0,$$

^{7.4}Since the thermal conductivity k is constant, h must not be a function of x if the expression as a whole cannot be a function of x .

which leads us to

$$\frac{\partial T(r, x)}{\partial x} = \frac{dT_s(x)}{dx},$$

where we have changed ∂ to a plain derivative, since T_s is only a function of x . In light of Eq. (7.12), we now conclude

$$(7.13) \quad \frac{\partial T(r, x)}{\partial x} = \frac{dT_m}{dx}.$$

In other words, the streamwise temperature gradient anywhere in the cross-section is equal to the streamwise gradient of the mean temperature. This result will be used in solving for the temperature distribution $T(r, x)$ in the pipe and, ultimately, the Nusselt number for laminar fully developed flow.

I&D Ex. 8.1
pp 475

We now address the actual problem of laminar fully developed pipe flow having a constant applied heat flux q_s'' at the pipe surface $r = r_0$. We assume constant properties and no energy generation. Derivation of the convection energy equation in cylindrical coordinates is beyond our present scope, but it is shown in numerous reference texts (e.g. Bejan, 1984, Chapter 1). For steady flow in (x, θ, r) coordinates (x is in the streamwise axial direction), we have

$$(7.14) \quad u \frac{\partial T}{\partial x} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + w_r \frac{\partial T}{\partial r} = \alpha \left[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right],$$

where u is again the axial streamwise velocity component, and v_θ and w_r are the azimuthal and radial velocity components. According to our hydrodynamic solution, $u = u(r)$ is given by Eq. (7.3), while $v_\theta = w_r = 0$. Also, the problem is symmetric, so that $\partial/\partial\theta = 0$. We will further assume^{7.5} that the mean temperature T_m is a linear function of x , so that $\partial T/\partial x$ is a constant in Eq. (7.13) and therefore $\partial^2 T/\partial x^2 = 0$. Eq. (7.14) then simplifies to

$$(7.15) \quad u(r) \frac{dT_m}{dx} = \alpha \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \right].$$

If we examine this equation more closely, we notice that since dT_m/dx is constant, the equation can be written as an ordinary differential equation

$$(7.16) \quad u(r) \frac{dT_m}{dx} = \alpha \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dT}{dr} \right) \right].$$

Recasting this equation in dimensionless temperature θ , as defined by Eq. (7.11), we write

$$(7.17) \quad u(r) \frac{dT_m}{dx} = \alpha (T_m - T_s) \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\theta}{dr} \right) \right],$$

which can be re-arranged as

$$(7.18) \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{d\theta}{dr} \right) = \frac{u(r)}{\alpha (T_m - T_s)} \frac{dT_m}{dx}.$$

^{7.5}This can be shown according to an energy balance (e.g. Incropera and Dewitt, 2002, §8.3.1).

Using the solution for $u(r)$ expressed in terms of the average velocity \bar{u} in Eq. (7.6), we can write

$$(7.19) \quad \frac{d}{dr} \left(r \frac{d\theta}{dr} \right) = A r \left[1 - \left(\frac{r}{r_0} \right)^2 \right],$$

where A is the constant defined by

$$A = \frac{2 \bar{u}}{\alpha (T_m - T_s)} \frac{dT_m}{dx}.$$

Eq. (7.19) can be integrated once to get

$$r \frac{\partial \theta}{\partial r} = \frac{A r^2}{2} - \frac{A r^4}{4 r_0^2} + C_1$$

and then again to obtain

$$\theta = \frac{A r^2}{4} - \frac{A r^4}{16 r_0^2} + C_1 \ln r + C_2,$$

where C_1 and C_2 are constants of integration.

The appropriate boundary conditions are similar to the hydrodynamic problem. Temperature has a given value T_s at the wall, $r = r_0$, so that $\theta = 0$ at $r = r_0$. A boundary condition at $r = 0$ can be deduced from either of

- the temperature is well-behaved, so that the natural log term must remain finite at $r = 0$
- because of symmetry of the overall problem, the temperature profile must also be symmetric about $r = 0$

Both these observations yield the same result that $C_1 = 0$. Applying the remaining condition, we find

$$C_2 = -\frac{3}{16} A r_0^2,$$

which enables us to write θ as

$$\theta = \frac{A r^2}{4} - \frac{A r^4}{16 r_0^2} - \frac{3}{16} A r_0^2.$$

A equivalent, but more convenient form is

$$(7.20) \quad \theta = -A r_0^2 \left[\frac{3}{16} + \frac{1}{16} \left(\frac{r}{r_0} \right)^4 - \frac{1}{4} \left(\frac{r}{r_0} \right)^2 \right].$$

The unknown value of A can be determined by employing the definition of the bulk mean temperature in Eq. (7.9). Let us write this in dimensionless form by forming θ on both sides (subtract T_s , then divide by $T_m - T_s$), which yields

$$(7.21) \quad \frac{T_m - T_s}{T_m - T_s} = \frac{2}{\bar{u} r_0^2} \int_0^{r_0} u \frac{T - T_s}{T_m - T_s} r dr.$$

Converting to dimensionless notation, this equation is

$$(7.22) \quad 1 = \frac{2}{\bar{u} r_0^2} \int_0^{r_0} u \theta r dr .$$

Substituting in for u and θ and integrating, we obtain

$$(7.23) \quad A r_0^2 = -\frac{96}{11} ,$$

as shown in Appendix B. Using this result, we can complete Eq. (7.20) as

$$(7.24) \quad \theta = \frac{96}{11} \left[\frac{3}{16} + \frac{1}{16} \left(\frac{r}{r_0} \right)^4 - \frac{1}{4} \left(\frac{r}{r_0} \right)^2 \right] .$$

This is the dimensionless temperature profile for the forced convection in a circular tube for constant wall heat flux.

Is it possible to develop this result even further? Fourier's Law written at the pipe's surface assumes the form

$$q_s'' = -k_f \left. \frac{\partial T}{\partial r} \right|_{r=r_0} .$$

Combining this with the expression for Newton's Law of Cooling in Eq. (7.10), we can express the convection coefficient as

$$h = -\frac{k_f}{T_m - T_s} \left. \frac{\partial T}{\partial r} \right|_{r=r_0} .$$

Let us now express h in terms of the dimensionless temperature θ from Eq. (7.11), where $T(r, x) = \theta [T_m(x) - T_s(x)] + T_s(x)$. We find

$$(7.25) \quad h = -k_f \left. \frac{\partial \theta}{\partial r} \right|_{r=r_0} .$$

Using the solution for θ in Eq. (7.24) to evaluate $\partial\theta/\partial r$ at $r = r_0$, we find

$$\left. \frac{\partial \theta}{\partial r} \right|_{r=r_0} = -\frac{96}{44 r_0} = -\frac{96}{22 D} = -\frac{48}{11 D} .$$

Substituting into Eq. (7.25) yields

$$h = k_f \frac{48}{11 D}$$

from which we can identify the Nusselt number as

$$(7.26) \quad Nu = \frac{h D}{k_f} = \frac{48}{11} \approx 4.364 .$$

The case of a constant surface temperature can be examined, however the mathematics of the problem are appreciably more difficult (Kays and Crawford, 1980, pp. 93–97). The result is similarly straightforward

$$(7.27) \quad Nu \approx 3.658 .$$

7.3. The Couette Problem

Let us now look at another internal configuration known as the Couette flow problem (Fig. 7.2). Here, two infinitely long flat plates a distance L

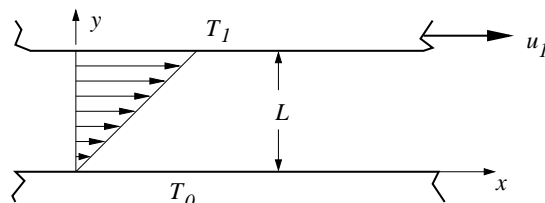


FIGURE 7.2. Schematic of the Couette flow problem.

apart enclose a working fluid characterized by constant density ρ , viscosity μ , and thermal conductivity k . One plate translates in its own plane past the other with a constant axial velocity u_1 . By convention, we specify the top plate as the moving one ($y = L$) and anchor our coordinate origin to the bottom plate at $y = 0$. The top plate is at a temperature of T_1 , while the bottom plate has a temperature of T_0 . As with the pipe problem, we assume the flow is fully developed, i.e. $\partial/\partial x = 0$, however, there is no pressure gradient. Rather, the top plate replaces pressure as the driving force. This configuration is a good model for a number of applications, for example a journal in an oil-filled bearing.

The problem can be framed in terms of the Navier–Stokes equations introduced in Eqs. (5.5) through (5.8) on pp. 57. For the fully developed flow assumption, we solve this problem as follows. The first term in the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

must vanish since the flow is fully developed, which leaves $\partial v/\partial y = 0$. At most, v could then be a function of x , but since the flow is fully developed v must be a constant. However, no-slip boundary conditions dictate $v = 0$ at both $y = 0$ and $y = L$, so the only admissible constant must be $v = 0$. As a consequence, the y -momentum equation, Eq. (5.7), vanishes altogether, while the only non-vanishing term in the x -momentum equation, Eq. (5.6), is $\partial^2 u/\partial y^2$. Clearly, since u is the only non-zero velocity component, and it is not a function of x (because of fully developed flow conditions), it must be the case that $u = u(y)$, so that Eq. (5.6) simplifies to the ordinary equation

$$\frac{d^2 u}{dy^2} = 0.$$

Integrating twice, we find

$$u(y) = C_1 y + C_2,$$

where C_1 and C_2 are evaluated from no-slip boundary conditions $u|_{y=0} = 0$ and $u|_{y=L} = u_1$. We find

$$u(y) = \frac{u_1 y}{L}.$$

With these results, the energy equation, Eq. (5.8) on pp. 57, simplifies to

$$\frac{d^2 T}{dy^2} = -\frac{\mu}{k} \left(\frac{du}{dy} \right)^2.$$

Notice that we wrote this as an ordinary equation rather than a partial one, since temperature is not a function of x , i.e. $T = T(y)$. The term on the right hand side represents viscous dissipation, which describes how mechanical energy is degraded to heat by the mechanism of friction. In the current problem, this can be viewed as a source term, because we know the solution for u and can evaluate this term explicitly. The result is

$$\frac{d^2 T}{dy^2} = -\frac{\mu u_1^2}{k L^2}.$$

Once again, we can perform straightforward integration to obtain the general solution

$$T(y) = -\frac{\mu u_1^2}{2k L^2} y^2 + C_3 y + C_4.$$

The boundary conditions $T|_{y=0} = T_0$ and $T|_{y=L} = T_1$ are used to determine constants C_3 and C_4 , so that the exact solution is

$$(7.28) \quad T(y) - T_0 = \frac{y}{L} \left[(T_1 - T_0) + \frac{\mu u_1^2}{2k} \left(1 - \frac{y}{L} \right) \right].$$

First, let us examine the case of $T_0 \neq T_1$. We will assume $T_1 > T_0$, although the opposite case would yield analogous results. Dividing by $T_1 - T_0$, we obtain

$$\frac{T(y) - T_0}{T_1 - T_0} = \frac{y}{L} \left[1 + \frac{1}{2} \frac{\mu u_1^2}{k (T_1 - T_0)} \left(1 - \frac{y}{L} \right) \right],$$

or, written in dimensionless terms

$$(7.29) \quad \theta = y^* \left[1 + \frac{Pr Ec}{2} (1 - y^*) \right],$$

where θ is dimensionless temperature, $y^* = y/L$ is the dimensionless coordinate,

$$Pr = \frac{c_p \mu}{k} = \frac{c_p \mu \rho}{k \rho} = \frac{1}{\alpha} \frac{\mu}{\rho} = \frac{\nu}{\alpha}$$

is the Prandtl number, and

$$Ec = \frac{u_1^2}{c_p (T_1 - T_0)}$$

is the Eckert number.

Note that the case of $Pr Ec = 0$ corresponds to the no-flow condition, and thus pure conduction. Thus, the temperature profile is a straight line

connecting T_0 and T_1 (Fig. 7.3). For other cases, viscous dissipation distorts the straight-line relationship as shown by plotting θ versus y^* in Fig. 7.3. An interesting aspect of this problem is the heat flux at the top wall, defined

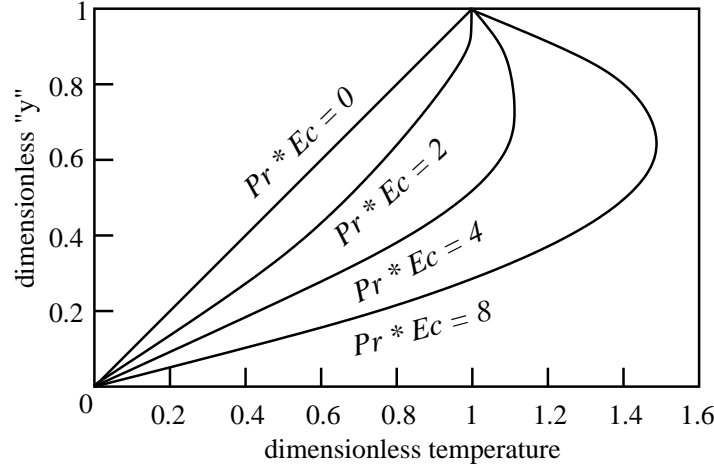


FIGURE 7.3. Temperature distribution in Couette flow with viscous dissipation for $T_1 > T_0$.

by

$$q = -k \left. \frac{dT}{dy} \right|_{y=L}.$$

From Eq. (7.28), we evaluate the derivative to obtain

$$\begin{aligned} \left. \frac{dT}{dy} \right|_{y=L} &= \left[\frac{T_1 - T_0}{L} + \frac{\mu u_1^2}{2k} \left(\frac{1}{L} - \frac{2y}{L^2} \right) \right] \Big|_{y=L} \\ &= \frac{T_1 - T_0}{L} - \frac{\mu u_1^2}{2kL} \\ &= \frac{T_1 - T_0}{L} \left[1 - \frac{\mu u_1^2}{2k(T_1 - T_0)} \right] \\ &= \frac{T_1 - T_0}{L} \left(1 - \frac{Pr Ec}{2} \right), \end{aligned}$$

which means that the heat flux is

$$(7.30) \quad q = -k \frac{T_1 - T_0}{L} \left(1 - \frac{Pr Ec}{2} \right).$$

There are clearly three cases of interest, depending upon the sign of the $(1 - Pr Ec/2)$ term (again, assuming $T_1 > T_0$):

- $Pr Ec < 2$, which makes the right hand side negative and therefore heat flows in the $-y$ direction, or from the upper wall into the liquid. This is what we would intuitively expect, given $T_1 > T_0$.

- $Pr Ec > 2$, which makes the right hand side positive and therefore heat flows in the $+y$ direction, or from the liquid into the wall, even though the upper wall is at a higher temperature than the lower wall. Not intuitive.
- $Pr Ec = 2$, which makes the right hand side vanish and thus there is no heat transfer at the upper wall. The upper wall behaves as an insulated boundary ($q'' = 0$), even though there is no actual insulation. Also not intuitive.

The opposite case from the above is $T_0 = T_1$. Here, Eq. (7.28) reduces to

$$T(y) - T_0 = \frac{\mu u_1^2}{2k} \frac{y}{L} \left(1 - \frac{y}{L}\right).$$

Non-trivial temperature distributions are due strictly to viscous dissipation effects. This configuration is symmetric about $y = L/2$. In fact, the maximum temperature occurs at the middle of the channel and can be computed by setting $y = L/2$ to get

$$T_{max} - T_0 = \frac{\mu u_1^2}{8k}.$$

Though not identical, this case is qualitatively similar to heat generation for conduction heat transfer in §3.4.

7.4. Empirical Correlations

Other problems related to internal convection are more complicated than our present level of theory will allow us to handle. As with external configurations, we must defer to empirically-based correlations using the dimensionless parameters we have developed.

7.4.1. Turbulent Flow in Circular Tubes. If the flow is fully developed, both hydrodynamically and thermally, the Dittus–Boelter correlation can be applied

$$(7.31) \quad Nu_D = 0.023 Re_D^{4/5} Pr^n,$$

where $n = 0.4$ for $T_s > T_m$ (heating) and $n = 0.3$ for $T_s < T_m$ (cooling). The experimentally established range of applicability according to Incropera and Dewitt (2002) is $0.7 \leq Pr \leq 160$, $Re_D > 10^4$, and $x > 10 D$. All properties are evaluated at T_m .

Eq. (7.31) is appropriate when T_s is not too different from T_m , however, if there is appreciable variation, the Sieder–Tate correlation should be used

$$(7.32) \quad Nu_D = 0.027 Re_D^{4/5} Pr^{1/3} \left(\frac{\mu}{\mu_s}\right)^{0.14}.$$

The limits are identical to Eq. (7.31), except that the Prandtl number can be extended up to $Pr \approx 16,700$. For Eq. (7.32), all properties are evaluated at T_m , except μ_s , which is evaluated at T_s . Eqs. (7.31) and (7.32) can be

reasonably applied for both the constant temperature and constant heat flux boundary conditions.

It is emphasized that both Eqs. (7.31) and (7.32) should only be used for fully turbulent problems. The transition regime between laminar and fully turbulent flow, $2000 < Re_D < 10^4$, entails additional considerations. Incropera and Dewitt (2002, §8.5) discuss additional results for these cases.

7.4.2. Non-Circular Tubes. Many engineering applications involve tubes or ducts having non-circular cross sections. Even for laminar flow, theoretical approaches are beyond the mathematics we have discussed thus far. For example, for a rectangular cross section, the momentum equation cannot be reduced to an ordinary differential equation like what we obtained for circular pipes in Eq. (7.1). However, results we have obtained for circular tubes can be applied as an initial approximation to non-circular cross sections using the *hydraulic diameter* as a length scale

$$(7.33) \quad D_h = \frac{4 A_c}{P},$$

where A_c is the cross-sectional area of the conduit and P is the wetted perimeter^{7.6}. The value of D_h can be computed directly from geometry, for example, for a rectangular duct of height a and width b , we obtain

$$(7.34) \quad D_h = \frac{4 a b}{2(a+b)} = \frac{2 a b}{a+b},$$

Thus, D_h should be used to calculate the non-circular analogs of the dimensionless parameters, e.g. Re_{D_h} and Nu_{D_h} .

For fully turbulent flows, the Dittus-Boelter and Sieder-Tate correlations can be reasonably used in conjunction with the hydraulic diameter. Results are less accurate for laminar flows, especially in geometries having sharp corners. Here, Nusselt numbers based on exact solutions should be used. For example, Table 7.1 shows selected results for channels of rectangular cross section arranged according to the aspect ratio $\phi = b/a$, where $b > a$. Kays and Crawford (1980) consider such problems in greater detail.

TABLE 7.1. Nusselt numbers for rectangular channels of various aspect ratios (Kays and Crawford, 1980).

ϕ	Nu_{D_h} (const q_s'')	Nu_{D_h} (const T_s)
1	3.61	2.98
2	4.12	3.39
3	4.79	3.96
4	5.33	4.44

^{7.6}Eq. (7.33) contains a factor of 4 so that the hydraulic diameter of a circular pipe is equal to its geometric diameter, i.e. $D_h = 4(\pi D^2/4)/(\pi D) = D$.

CHAPTER 8

Natural Convection

Up until now we have considered cases where convection is a result of the forced motion of a fluid, e.g. a pressure gradient, moving boundary, etc. However, there are many cases where initial heat transfer causes the local temperature of a fluid to change and thus its density to change as well. This causes a “natural” fluid motion which in turn gives rise to natural, or free convection. Thus, free convection arises because of buoyancy forces. A few notable situations are refrigerator coils, piping, electronics, steam radiators in old homes, and flow along window panes.

Generally, the density of a fluid decreases with increasing temperature because of volumetric expansion, which *may* induce a natural convection flow. This actually depends upon the configuration of the problem. In particular, an *instability* must be able to occur. For example, imagine 2 infinite parallel horizontal plates which are both fixed. If the top plate is hotter than the lower one (Fig. 8.1, left), the configuration is inherently stable because the lighter fluid is already above the cooler heavier fluid. There is no

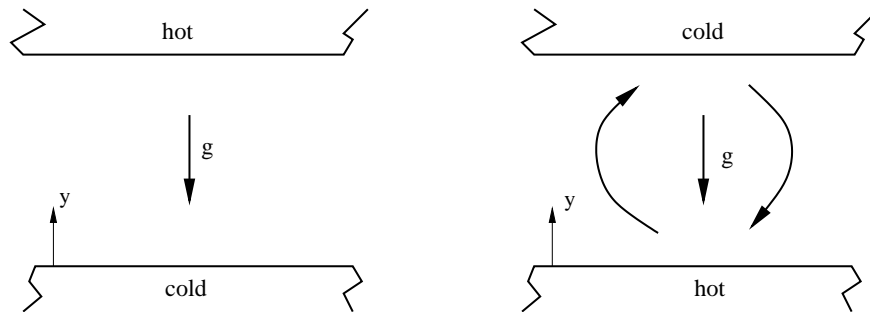


FIGURE 8.1. *The concept of instability as a necessary condition for natural convection.*

tendency for the system to move away from this state of equilibrium and any heat transfer between the plates will be a result purely of conduction. However, if the bottom wall is hotter (Fig. 8.1, right), the hot light fluid will initially be below the cooler heavier fluid. This “top heavy” system will tend to flow, with the heavy fluid falling by the action of gravity and the light fluid correspondingly rising. A circulation pattern will arise, so that heat transfer between the plates is effected primarily by natural convection.

8.1. Wall Bounded Convection on a Vertical Flat Surface

Let us start by considering the familiar laminar boundary layer situation, except this time the flat plate is placed vertically. If we consider the problem first on a conceptual basis, we quickly find that the boundary layer velocity profile will be fundamentally different from the standard case we are accustomed to, i.e. as depicted in Fig. 5.1. While we still have a no-slip boundary condition on the plate itself, the fluid far away from the plate ($y \rightarrow \infty$) is at rest, rather than moving at some non-zero freestream velocity. If we assume that the plate temperature T_s is greater than the fluid temperature T_∞ , lighter fluid will rise, causing an upward convection pattern. The resulting velocity profile would look qualitatively like the one plotted in Fig. 8.2.

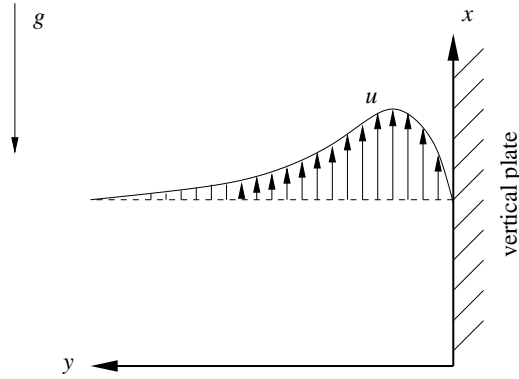


FIGURE 8.2. *Boundary layer profile on a vertical flat plate.*

We are again assuming incompressible flow with one important exception: we must allow for the buoyancy force which drives the flow. In this term, the density is not constant! Using the boundary layer approximation, we write the resulting equation in the streamwise direction x as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dP}{dx} - g + \nu \frac{\partial^2 u}{\partial y^2},$$

where g represents the gravity vector. There is no body force in the y direction (i.e. horizontally), therefore the x pressure gradient at any point within the boundary layer is the same as that in the quiescent region far outside the boundary layer. Since there is no fluid motion in this area, vertical pressure gradient is prescribed simply by hydrostatic considerations, $\partial P/\partial x = -\rho_\infty g$, where ρ_∞ is the density of quiescent fluid far from the plate. Substituting back into the equation gives

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{g}{\rho} (\rho_\infty - \rho) + \nu \frac{\partial^2 u}{\partial y^2}.$$

The first term on the right hand side is the buoyancy force and in this case it is the driving force of the flow. If temperature is the only factor affecting density^{8.1}, this term can be described by a thermodynamic property known as the *volumetric thermal expansion coefficient*

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T} \right) \Big|_P,$$

which describes the change of density as a function of temperature at constant pressure. This can be approximated by a simple finite difference expression

$$\beta \approx -\frac{1}{\rho} \frac{\rho_\infty - \rho}{T_\infty - T},$$

which implies

$$\rho_\infty - \rho \approx \rho \beta (T - T_\infty).$$

This idealization is the so-called *Boussinesq approximation* for density variation. Substituting, the streamwise momentum equation becomes

$$(8.1) \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = g \beta (T - T_\infty) + \nu \frac{\partial^2 u}{\partial y^2},$$

which nicely shows how the driving buoyancy force is related to a temperature difference.

Buoyancy effects are strictly limited to the momentum equation, therefore the remaining equations are unaffected. Specifically, for conservation of mass, we have the usual

$$(8.2) \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and for conservation of energy, we likewise have

$$(8.3) \quad u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}.$$

Eqs. (8.1) through (8.3) are the boundary layer form of the natural convection equations. Viscous dissipation has been neglected here, primarily due to the small velocities associated with natural convection.

Examining these equations, we discover a major mathematical complication as compared to forced convection: T now appears in the momentum equation in addition to u appearing in the energy equation. Therefore, this system is fully coupled, as opposed to forced convection, which was uncoupled since T only appeared in the energy equation. We can no longer treat the hydrodynamic problem independently. The equations must be solved simultaneously.

^{8.1}Density changes can arise via other means, as in for example supersonic flow. We do not consider such cases here.

The thermal expansion coefficient β is a material property. For an ideal gas, the Ideal Gas Law $\rho = P/RT$ gives $\partial\rho/\partial T = -P/RT^2$, so that

$$\beta = -\frac{1}{\rho} \left(\frac{\partial\rho}{\partial T} \right) = \frac{1}{\rho} \frac{P}{RT^2} = \frac{RT}{P} \frac{P}{RT^2} = \frac{1}{T}.$$

For liquids and non-ideal gases, β must be obtained from appropriate tables.

8.2. Dimensionless Formulation

Let us now look at some of the non-dimensional parameters that are relevant in natural convection. We introduce the usual scales, e.g. similar to those described in §5.3

$$\begin{aligned} x^* &= \frac{x}{L}, & y^* &= \frac{y}{L}, \\ u^* &= \frac{u}{u_0}, & v^* &= \frac{v}{u_0}, & T^* &= \frac{T - T_\infty}{T_s - T_\infty}, \end{aligned}$$

where L is a characteristic length and u_0 is a characteristic velocity^{8.2}. Non-dimensionalizing the equations according to the results we developed in §4.4, we find, as usual, that the continuity equation remains the same in non-dimensional form

$$(8.4) \quad \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0.$$

The momentum and energy equations become

$$(8.5) \quad u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{g \beta (T_s - T_\infty) L}{u_0^2} T^* + \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}},$$

and

$$(8.6) \quad u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re Pr} \frac{\partial^2 T^*}{\partial y^{*2}}.$$

There is a new coefficient we have not seen before in front of the buoyancy term. However, we recall that u_0 is not a readily prescribed scale, so we can arbitrarily multiply by another dimensionless parameter to obtain a new dimensionless parameter^{8.3}. In this case, we multiply this quantity by the square of the Reynolds number, $Re = u_0 L/\nu$, to get the *Grashof number*

$$Gr = \frac{g \beta (T_s - T_\infty) L}{u_0^2} \times \frac{u_0^2 L^2}{\nu^2},$$

which gives

$$(8.7) \quad Gr = \frac{g \beta (T_s - T_\infty) L^3}{\nu^2}.$$

^{8.2}Characteristic velocity is not readily identifiable since conditions outside the boundary layer are at rest and thus $u_\infty = 0$. This is slightly different from the method of non-dimensionalization shown in §5.3.

^{8.3}Recall that this is permissible according to the Buckingham-Pi theorem from fluid mechanics.

The Grashof number plays the same role in free convection that the Reynolds number plays in forced convection, i.e. Re is the ratio of inertial forces to viscous forces, whereas Gr is the ratio of buoyancy forces to viscous forces. We can then recast momentum conservation in Eq. (8.5) as

$$(8.8) \quad u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = Gr T^* + \frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}},$$

The three conservation laws, Eqs. (8.4), (8.6), and (8.8), are functions of the dimensionless parameters Re , Pr , and Gr . We would therefore expect solutions of the natural convection problem, given by the Nusselt number, to be of the form

$$Nu = Nu(Re, Pr, Gr) .$$

There is a subtle aspect to this interpretation. In formulating the Grashof number, we were able to eliminate the velocity scale u_0 based on the observation that we could not necessarily characterize it appropriately. Yet, it could be that an external forcing results in an explicit u_0 , so that a problem would be a mixture of forced convection and natural convection. The above presumption about the functional dependence of Nu is valid when forced and natural convection aspects are comparable, i.e.

$$\frac{Gr}{Re^2} \sim 1 .$$

Looking back at the form of Gr in Eq. (8.5), this condition is equivalent to

$$g \beta (T_s - T_\infty) L \sim u_0^2 .$$

Consequently, if forced convection dominates a problem,

$$\frac{Gr}{Re^2} \ll 1 \quad \text{and} \quad Nu = Nu(Re, Pr) .$$

That is, free convection effects are so small as to be neglected. Conversely, if natural convection is dominant, we have

$$\frac{Gr}{Re^2} \gg 1 \quad \text{and} \quad Nu = Nu(Pr, Gr) .$$

Here, the small forced convection effect would be neglected, although strictly speaking, natural convection occurs exclusively only for $Gr/Re^2 \rightarrow \infty$, for which $u_0 = 0$.

8.3. Similarity Solution for the Vertical Plate

As already noted, natural convection is a difficult problem to solve analytically because the equations are fully coupled. However, as with external forced convection on the horizontal flat plate, there are no obvious scales for the problem, so we suspect that a similarity transform^{8.4} might be used. If successful, this would reduce the partial differential problem to an ordinary differential problem.

^{8.4}The similarity technique was introduced in §4.6.

From Fig. 8.2, we deduce the boundary conditions

$$y = 0 : \quad u = v = 0, \quad T = T_s$$

and

$$y \rightarrow \infty : \quad u \rightarrow 0, \quad T \rightarrow T_\infty,$$

where T_s is the temperature of the plate. In this case, the transformation variables

$$\eta = \frac{y}{x} \left(\frac{Gr_x}{4} \right)^{1/4} \quad \text{and}$$

$$f(\eta) = \frac{\psi}{4\nu} \left(\frac{4}{Gr_x} \right)^{1/4}$$

are appropriate, where $\psi = \psi(x, y)$ is the stream function defined in Eq. (6.5) on pp. 67 and

$$Gr_x = \frac{g\beta(T_s - T_\infty)x^3}{\nu^2}$$

is the local Grashof number based on x .

Transforming the momentum equation appropriately, we find it becomes

$$f''' + 3ff'' - 2(f')^2 + T^* = 0,$$

where the prime symbol indicates the regular derivative of $f(\eta)$. The energy equation becomes

$$T^{*''} + 3Pr f T^{*'} = 0.$$

Like the horizontal flat plate problem, the momentum equation remains non-linear. Moreover, the two equations are still coupled to each other. i.e. f and T^* appear in both. In this case, a numerical method is typically invoked to complete the solution.

It can be shown (e.g. Incropera and Dewitt, 2002, pp. 541–542) that the general form for the local Nusselt number is

$$(8.9) \quad Nu_x = \frac{hx}{k} = \left(\frac{Gr_x}{4} \right)^{1/4} F(Pr),$$

where $F(Pr)$ is a function of the Prandtl number as determined by the solution of the above problem. A correlation, purported to be accurate to within about 0.5 % (Incropera and Dewitt, 2002, pp. 542) is given by the expression

$$(8.10) \quad F(Pr) = \frac{0.75 Pr^{1/2}}{(0.609 + 1.221 Pr^{1/2} + 1.238 Pr)^{1/4}}.$$

The result can also be used to calculate the overall (averaged) Nusselt number according to the definition given by Eq. (5.2) on pp. 54 as

$$\bar{h} = \frac{1}{L} \int_0^L h dx = \frac{1}{L} \int_0^L \frac{Nu_x k}{x} dx = \frac{1}{L} \int_0^L \frac{k}{x} \left(\frac{Gr_x}{4} \right)^{1/4} F(Pr) dx.$$

Substituting the local Grashof number, and observing that a number of factors in the argument, including $F(Pr)$ are not functions of x , and so can be taken outside the integral, we find

$$\begin{aligned}
 \bar{h} &= \frac{F(Pr) k}{4^{1/4} L} \int_0^L \frac{1}{x} \left(\frac{g \beta (T_s - T_\infty) x^3}{\nu^2} \right)^{1/4} dx \\
 &= \frac{F(Pr) k}{4^{1/4} L} \left(\frac{g \beta (T_s - T_\infty)}{\nu^2} \right)^{1/4} \int_0^L \frac{1}{x} x^{3/4} dx \\
 &= \frac{F(Pr) k}{L} \left(\frac{g \beta (T_s - T_\infty)}{4 \nu^2} \right)^{1/4} \int_0^L \frac{1}{x^{1/4}} dx \\
 &= \frac{F(Pr) k}{L} \left(\frac{g \beta (T_s - T_\infty)}{4 \nu^2} \right)^{1/4} \frac{4}{3} x^{3/4} \Big|_0^L \\
 &= \frac{4}{3} \frac{F(Pr) k}{L} \left(\frac{g \beta (T_s - T_\infty) L^3}{4 \nu^2} \right)^{1/4} \\
 &= \frac{4}{3} \frac{k}{L} \left(\frac{Gr_L}{4} \right)^{1/4} F(Pr) \\
 &= \frac{4}{3} \frac{k}{L} Nu_L,
 \end{aligned}$$

where the last line results directly from the form of the solution given in Eq. (8.9). Moving k and L over to the left hand side, we find the average Nusselt number to be

$$(8.11) \quad \overline{Nu_L} = \frac{\bar{h} L}{k} = \frac{4}{3} Nu_L,$$

where Nu_L is the local Nusselt number at the end of the plate, i.e. at $x = L$.

As with forced flow, turbulent effects can arise in natural convection. We mentioned above that the Grashof number, defined by Eq. (8.7), can be taken as the ratio of buoyancy effects versus viscous effects. The interplay between these two phenomena is very similar to that defined by the Reynolds number for forced convection, i.e. viscous forces tend to dissipate disturbances which could grow large enough to de-stabilize the flow. We would therefore expect that Gr serves as a direct measure of the tendency toward turbulent flow for natural convection in the same sense as Re for forced convection. However, data correlate more strongly to a closely-related parameter call the *Rayleigh number*, which is simply the product of the Grashof and Prandtl numbers. The critical Rayleigh number for turbulent flow is approximately.

$$(8.12) \quad Ra_x = Gr_x Pr = \frac{g \beta (T_s - T_\infty) x^3}{\nu \alpha} \approx 10^9.$$

Gebhart et al. (1988) discuss this in greater detail.

I&D Ex. 9.1
pp 543

8.4. Empirical Correlations

There is little more than can be done from an analytical standpoint with natural convection because of the complexity of the problem, so we tend to focus more on empirically correlated results. As with turbulent flow, most of the data correlate more appropriately to Ra rather than Gr . Properties are again evaluated at the film temperature, defined by Eq. (6.1) on pp. 66.

8.4.1. Vertical Plate. The Churchill–Chu correlation can be applied over the entire range of Ra

$$(8.13) \quad \overline{Nu}_L = \left[0.825 + \frac{0.387 Ra_L^{1/6}}{\left(1 + [0.492/Pr]^{9/16}\right)^{8/27}} \right]^2,$$

although somewhat better accuracy for strictly laminar flow is given by

$$(8.14) \quad \overline{Nu}_L = 0.68 + \frac{0.67 Ra_L^{1/4}}{\left(1 + [0.492/Pr]^{9/16}\right)^{4/9}} \quad \text{where } Ra_L < 10^9.$$

These correlations are applicable to cases where the plate temperature T_s is constant. Incropera and Dewitt (2002, pp. 546) describe procedures to extrapolate these results to several other configurations.

I&D Ex. 9.2
pp 546

8.4.2. Horizontal Cylinder. Heated horizontal cylinders are closely related to the vertical plate configuration, except that the boundary layer curves around the contour of the body rather than being strictly vertical (Fig. 8.3). Development begins on the underside of the cylinder and the boundary layer eventually detaches in the form of a plume that rises above the cylinder. Churchill and Chu have formulated a correlation valid over a

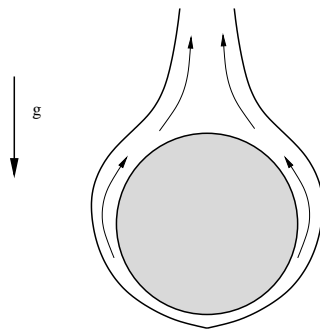


FIGURE 8.3. *Natural convection boundary layer on a horizontal cylinder.*

wide range of Ra

$$(8.15) \quad \overline{Nu}_L = \left[0.6 + \frac{0.387 Ra_D^{1/6}}{\left(1 + [0.559/Pr]^{9/16}\right)^{8/27}} \right]^2, \quad Ra_D < 10^{12}.$$

where the Rayleigh number is based on the pipe diameter D .

8.4.3. Rectangular Enclosures. Configurations we have discussed thus far are primarily of the external flow type, but internal configurations also arise in applications, for example in the form of various enclosures (Fig. 8.4). Here, the bottom and top surfaces are adiabatic, while the ver-

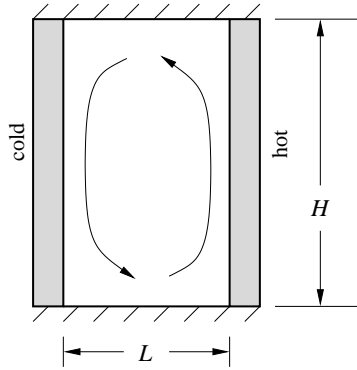


FIGURE 8.4. *Recirculating natural convection flow pattern in a vertically-oriented enclosure. This particular configuration is an excellent model for analyzing double-pane windows.*

tical surfaces have specific temperatures, T_1 on the hot side and T_2 on the cold side, and are separated by a distance L . If the Rayleigh number is sufficiently high, a recirculating flow pattern will form, similar to that shown in the figure. For small Ra_L , the flow is weak and heat transfer occurs mainly by conduction, i.e. $Nu_L \sim 1$. For larger Ra_L , a number of correlations have been devised, e.g.

$$(8.16) \quad \overline{Nu}_L = 0.22 \left(\frac{Pr Ra_L}{0.2 + Pr} \right)^{0.28} \left(\frac{H}{L} \right)^{-0.25}, \quad 10^3 < Ra_L < 10^{10},$$

which is valid for aspect ratios $2 < H/L < 10$ and $Pr < 10^5$. Other correlations are available for larger aspect ratios (Özişik, 1985; Incropera and Dewitt, 2002).

CHAPTER 9

Heat Exchangers

Heat exchangers are devices that facilitate heat transfer between two fluids at different temperatures without allowing them to mix. Usually, the fluids are separated by a solid boundary, for example as in car radiators, air conditioners, distillers, etc. These are called indirect contact exchangers. However, if the fluids do not tend to mix naturally, a direct contact heat exchanger may be used, e.g. a water chiller.

Heat exchangers are typically classified by flow arrangement and type of construction. In parallel flow units, fluids enter at the same end and flow in the same direction, while in counter-flow, the flow of the two fluids is in opposite directions. Fig. 9.1 illustrates these arrangements for a simple heat exchanger consisting of concentric tubes.

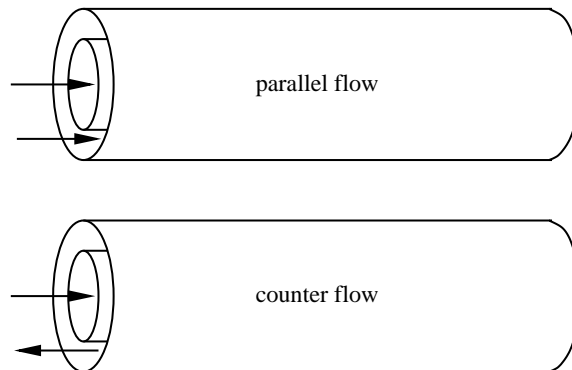


FIGURE 9.1. *Flow arrangements for concentric tube heat exchanger. One fluid flows within the innermost tube, while the other flows in the gap between the two tubes.*

The concentric tube heat exchanger is a good model for introducing a number of concepts. According to convention, the inner tube is called the “tube”, while the outer tube is referred to as the “shell”. Thus, Fig. 9.1 shows the simplest *shell-and-tube heat exchanger* architecture. However, according to our results for fins in §3.5 we suspect that actual units might not be as simple as that depicted in Fig. 9.1 because of the limited surface area of the tube. Instead, units routinely utilize many tubes within a single shell, so that the area for heat transfer is maximized (Fig. 9.2). Variations

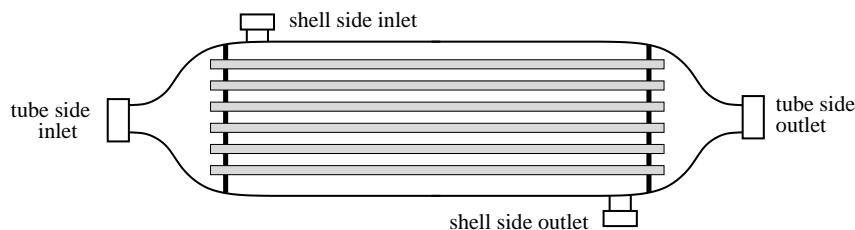


FIGURE 9.2. *Shell-and-tube heat exchanger with one tube pass and one shell pass. Tube side enters and exits through manifolds.*

on this basic architecture are shown in reference texts (e.g. Özişik, 1985; Mills, 1999; Incropera and Dewitt, 2002).

It is also common to classify heat exchangers according to the area available for heat transfer per unit volume of the unit, i.e.

$$\varphi = \frac{\text{total area available for heat transfer}}{\text{total volume of heat exchanger}}.$$

Units having $\varphi > 700 \text{ m}^{-1}$ are arbitrarily referred to as being compact heat exchangers, regardless of their design architecture. Table 9.1 summarizes φ for several classes of heat exchangers.

TABLE 9.1. Approximate area density for various classes of heat exchangers (Özişik, 1985).

Type	$\varphi \text{ (m}^{-1}\text{)}$
industrial shell-and-tube	70 – 500
automotive radiators	1000
precision specialty units	5000 – 10,000
human lung	20,000

9.1. The Thumbnail Diagram

In analyzing heat exchangers, we will find the concept of the thumbnail diagram to be especially useful. The diagram is simply a plot of the temperatures of the two streams, hot side (H) and cold side (C), as functions of the distance from an origin. For example, Fig. 9.3 shows a simple tube-type unit of length L running in parallel flow mode. Assuming the origin to be the inlet side at $x = 0$, temperatures of the two streams are shown, along with their flow directions. Although direction arrows are not mandatory, they will be extremely helpful for the “bookkeeping” aspect of heat exchanger calculations.

Fig. 9.3 shows a number of interesting concepts for the parallel flow arrangement:

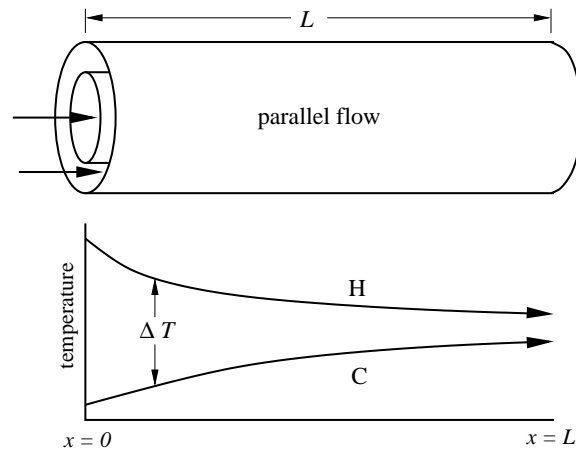


FIGURE 9.3. *Thumbnail diagram for a simple tube-type heat exchanger using the parallel flow arrangement.*

- The outlet temperature of the cool side cannot exceed that of the hot side. At most, they can be equal, otherwise an un-physical temperature gradient exists. Therefore, the effectiveness of parallel flow is limited and not typically used for heat recovery.
- The temperature gradient between the streams, ΔT , is not constant. Therefore the local heat flux q'' varies with x .
- If we assume that the wall temperature of the material separating the streams is the average between the two streams, then wall temperature is roughly constant. This might be an advantageous design aspect if the wall material is sensitive to stresses induced by temperature differences along x .

Fig. 9.4 shows thumbnail diagrams for three other cases: counter-flow, and condensing and boiling heat transfer. For the counter-flow design, the

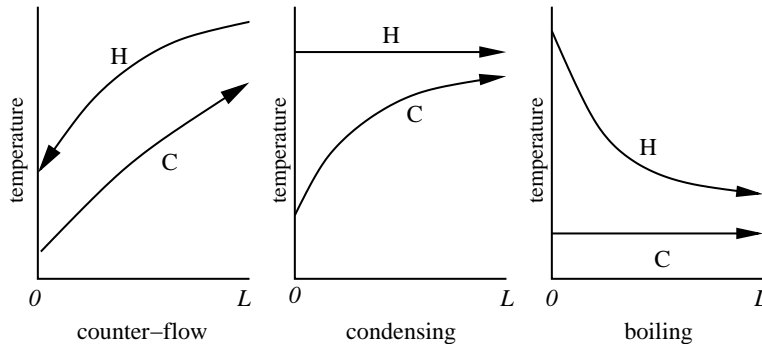


FIGURE 9.4. *Thumbnail diagrams for the counter-flow arrangement, and for condensing and boiling heat transfer.*

exit temperature of the cold fluid can in fact be higher than that of the hot fluid. This makes the counter-flow design more attractive than parallel flow units if the design requirements permit such a choice. Clearly, the wall temperature in these units is not constant, which might cause large thermal stresses that may eliminate certain materials from design consideration. Although not necessarily the case, ΔT might also be constant along x for the counter-flow arrangement.

Although we do not discuss the physics of condensation and boiling heat transfer *per se*, we can examine their application in heat exchangers utilizing the simple fact that temperature is constant during a phase change. It should be clear that the flow direction of the phase change stream is irrelevant because of its constant temperature. For the condenser, heat energy is liberated from the condensate and absorbed by the cold side, causing its temperature to rise^{9.1}. For a boiling heat exchanger, the reverse is true: the hot side transfers energy to boil the fluid on the cold side and cools in the process.

Thumbnail diagrams can be drawn for more complicated configurations as well. For example, Fig. 9.5 illustrates a shell-and-tube exchanger having one shell pass and two tube passes, along with its corresponding thumbnail diagram.

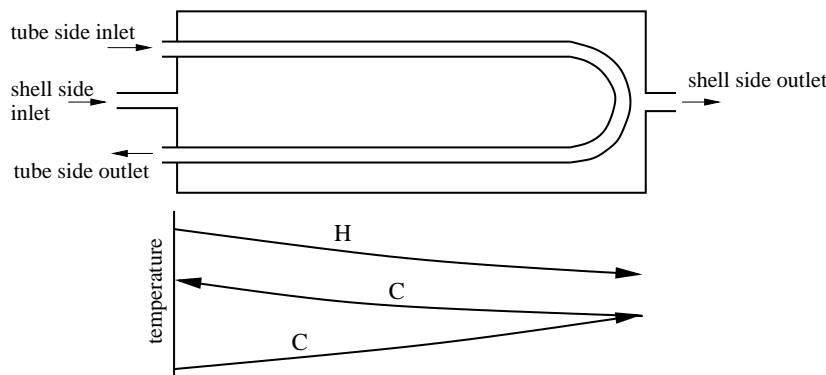


FIGURE 9.5. Thumbnail diagram for one shell pass, two tube pass heat exchanger. We have assumed the tube side contains the cold fluid.

9.2. Overall Heat Transfer Coefficient

We have devoted considerable attention to obtaining convection coefficients in Chapters 5 through 8. While these problems dealt strictly with single flows, the heat exchanger presents us with two flows to analyze. How

^{9.1}A cold can of beer on a warm humid day is good example of this phenomenon. Water vapor condenses on the outside of the can, transferring heat to the beer, causing it to warm up.

should this be handled? As we discussed in Chapter 3, the circuit analogy can be used to obtain overall effective thermal resistances. The same concept can be applied here to calculate an *overall heat transfer coefficient* for a heat exchanger. The general form of this coefficient is written^{9.2}

$$(9.1) \quad U = \frac{1}{A_t R_t},$$

where A_t is the total area available for heat transfer and R_t is the effective (overall) thermal resistance. Calculating U is clearly a matter of first determining R_t . For a simple shell-and-tube heat exchanger, thermal resistance is the sum of three basic resistive components in series (Fig. 9.6):

- convective resistance on the tube side
- conductive resistance of the wall between the two flow streams
- convective resistance on the shell side

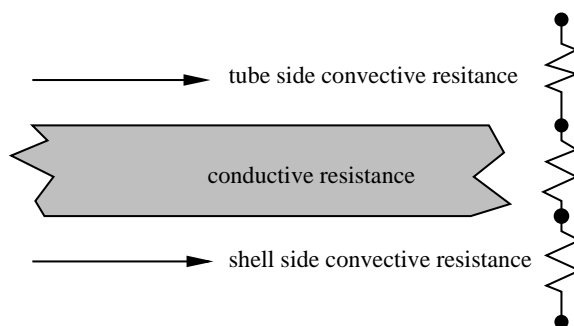


FIGURE 9.6. Resistive components involved in calculating the overall thermal resistance.

We can write the overall resistance as

$$(9.2) \quad R_t = \frac{1}{A_i h_i} + R_c + \frac{1}{A_o h_o},$$

where subscripts i and o designate inner and outer, respectively, and R_c is the conductive resistance for a cylindrical surface derived in Eq. (3.22) on pp. 22 as

$$R_c = \frac{\ln(r_o/r_i)}{2\pi L k}.$$

Notice that we can use either A_i or A_o as the total area available for heat transfer A_t in Eq. (9.1). It makes no difference which one is used, as long

^{9.2}By convention, U is the symbol for overall convection coefficient for heat exchangers, rather than some form of h which is what we might have expected. This notation is the standard (e.g. Özişik, 1985; Mills, 1999; Incropera and Dewitt, 2002).

as the basis is specified^{9.3}. Using $A_o = 2 \pi r_o L$ as the basis, we obtain

$$U_o = \frac{1}{A_o R_t} = \frac{1}{A_o \left(\frac{1}{A_i h_i} + \frac{\ln(r_o/r_i)}{2 \pi L k} + \frac{1}{A_o h_o} \right)},$$

which, after simplifying yields

$$(9.3) \quad U_o = \frac{1}{\frac{r_o}{r_i h_i} + \frac{r_o \ln(r_o/r_i)}{k} + \frac{1}{h_o}}.$$

Similarly, for $A_i = 2 \pi r_i L$ as the basis, we get

$$(9.4) \quad U_i = \frac{1}{\frac{1}{h_i} + \frac{r_i \ln(r_o/r_i)}{k} + \frac{r_i}{r_o h_o}}.$$

If the wall thickness is minimal^{9.4} and the thermal conductivity is high, tube wall resistance can be neglected to obtain

$$(9.5) \quad U = \frac{1}{\frac{1}{h_i} + \frac{1}{h_o}}.$$

In the course of normal operation, inner surfaces of a heat exchanger can become coated with deposits that leach out of the working fluids and/or corrode due to reaction with the fluid. These factors present additional resistance to heat transfer that can be modeled via a *fouling factor*. Eq. (9.2) can be modified as

$$(9.6) \quad R_t = \frac{F_i}{A_i} + \frac{1}{A_i h_i} + R_c + \frac{1}{A_o h_o} + \frac{F_o}{A_o}$$

to consider fouling, where F_o and F_i are the fouling factors for the outer and inner surfaces, respectively. Generally, performance is gradually degraded over time and costs are increased because of maintenance requirements and down time. The fouling factor is generally a known quantity based on the working fluid (Table 9.2).

9.3. LMTD Analysis

At this point we probably already suspect that the somewhat exacting analysis methods we have applied to previous problems will not be possible for heat exchangers. The flow physics of these units is simply too complex to obtain analytical solutions, e.g. because of turbulence, developing flow, separation, etc. Moreover, diverse geometry and architecture preclude generalizing results into a few relevant correlations. Instead, we will take what

^{9.3}Calculating the overall heat transfer will involve multiplying by A_t using a form of Newton's Law of Cooling (see Eq. (9.7)), so that, as long as consistency is preserved with respect to which area is used, the effect will cancel itself out.

^{9.4}If wall thickness is small, then $r_i \approx r_o$, which means that $\ln(r_o/r_i) = \ln(1) = 0$, resulting in a vanishing wall resistance term.

TABLE 9.2. Some representative fouling factors (TEMA, 1978).

Fluid	Fouling Factor ($m^2 K/W$)
seawater below $50^\circ C$	0.0001
seawater above $50^\circ C$	0.0002
fuel oil	0.0009
machine oils	0.00018
vegetable oils	0.00053
most refrigerants	0.0002
steam (non oil bearing)	0.0001
alcohol vapor	0.00009
industrial air	0.0004

may be thought of as an approximate integral approach in which the analysis is only dependent upon temperatures at the inlets and outlets and the overall convection coefficient.

Let us cast our analysis in the form of Newton's Law of Cooling

$$(9.7) \quad q = A_t U_m \Delta T_m,$$

where A_t and U_m are the total area available for heat transfer and the overall convection coefficient, respectively, and ΔT_m is a temperature difference that must be determined. In general, the temperature difference between the hot and cold fluids is not a constant, as suggested by Figs. 9.3 and 9.4. How, then do we determine ΔT_m ?

Here, we will introduce the method of the "Log Mean Temperature Difference" (LMTD) for solving heat exchanger problems. This procedure allows us to calculate ΔT_m , which can be thought of as an appropriately averaged temperature difference between the two flow streams. It is probably no surprise that ΔT_m depends upon the heat exchanger configuration, flow arrangement, etc. To illustrate the concept, we will develop ΔT_m for the simple tube-type unit shown in Fig. 9.3. There are a number of additional assumptions we must make to implement this method

- the unit is insulated such that no heat is exchanged with its surroundings; heat transfer only takes place between the hot and cold streams within the unit
- specific heats of both fluids are constant
- overall heat transfer coefficient is constant
- potential and kinetic energy changes can be neglected

We start by adding some detail to the thumbnail diagram (Fig. 9.7). As usual, let x be the distance along the heat exchanger, A be the area, \dot{m}_c and \dot{m}_h be the mass flow rates of the cold and hot sides, respectively, $\Delta T = T_h - T_c = \Delta T(x)$ be the local temperature difference between the hot and cold sides, and $U = U(x)$ be the local overall heat transfer coefficient. Each stream experiences a differential change in temperature as a result of

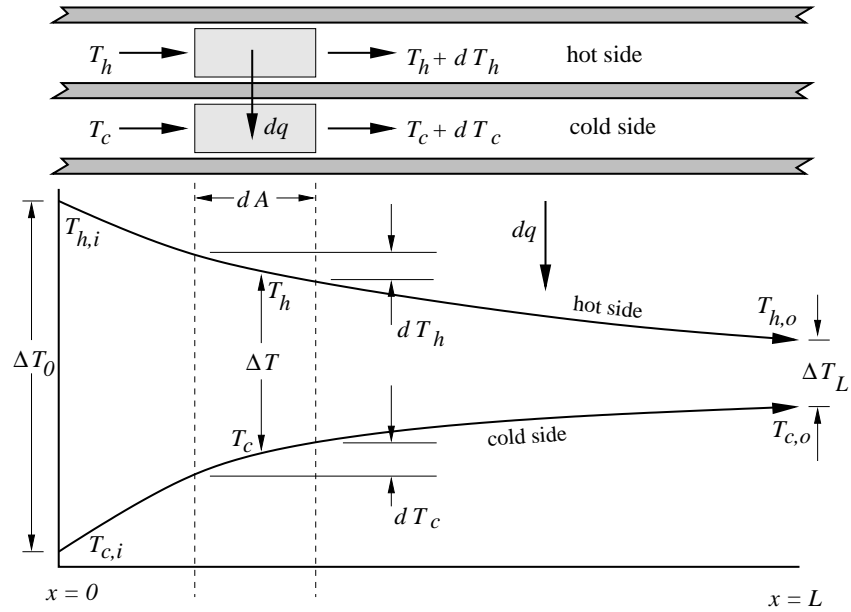


FIGURE 9.7. Differential analysis for LMTD calculation of the parallel flow simple tube-type heat exchanger.

interacting over an area dA . We can write a differential form of Newton's Law at x as

$$(9.8) \quad dq = dA U \Delta T,$$

where dq is the thermal energy transferred from the hot stream to the cold stream through area dA . The differential energy transfer can also be expressed in terms of the change of temperature of each stream over dA . Specifically, the heat lost by the hot side results from a change in temperature of $(T_h + dT_h) - T_h$ and is thus equal to

$$(9.9) \quad dq = -\dot{m}_h c_h dT_h,$$

while for the cold side we have

$$(9.10) \quad dq = \dot{m}_c c_c dT_c,$$

where c_h and c_c are the fluid specific heats. From the local temperature difference $\Delta T = T_h - T_c$, we see that $d(\Delta T) = dT_h - dT_c$, so that we can use Eqs. (9.9) and (9.10) to write

$$(9.11) \quad d(\Delta T) = -\frac{dq}{\dot{m}_h c_h} - \frac{dq}{\dot{m}_c c_c} = -dq \left(\frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c} \right).$$

Since the mass flow rates and specific heats are assumed to be constant, we let the last term in brackets be represented by the constant C_0 , i.e.

$$(9.12) \quad d(\Delta T) = -dq C_0.$$

Substituting dq from Eq. (9.8), we find

$$(9.13) \quad d(\Delta T) = -dA U \Delta T C_0,$$

or, dividing by ΔT ,

$$(9.14) \quad \frac{d(\Delta T)}{\Delta T} = -dA U C_0.$$

Eq. (9.14) can be integrated over the whole length of the heat exchanger

$$(9.15) \quad \int_{\Delta T_{x=0}}^{\Delta T_{x=L}} \frac{d(\Delta T)}{\Delta T} = -C_0 \int_0^{A_t} U dA.$$

Notice that C_0 comes outside the integral since it is a constant. The right hand side can be multiplied by A_t/A_t , i.e.

$$(9.16) \quad \int_{\Delta T_{x=0}}^{\Delta T_{x=L}} \frac{d(\Delta T)}{\Delta T} = -C_0 A_t \times \frac{1}{A_t} \int_0^{A_t} U dA.$$

Let us now define the average overall heat transfer coefficient for the whole heat exchanger in the usual fashion as

$$(9.17) \quad U_m = \frac{1}{A_t} \int_0^{A_t} U dA,$$

so that by integrating the left hand side of Eq. (9.16), we get

$$(9.18) \quad \left(\ln \Delta T \right) \Big|_{\Delta T_{x=0}}^{\Delta T_{x=L}} = \ln \left(\frac{\Delta T_{x=L}}{\Delta T_{x=0}} \right) = -U_m A_t C_0,$$

or rewriting to remove the minus sign, it becomes

$$(9.19) \quad \ln \left(\frac{\Delta T_{x=0}}{\Delta T_{x=L}} \right) = U_m A_t C_0.$$

Eq. (9.19) is an interesting intermediate result, but is not yet in the form of Newton's Law of Cooling in Eq. (9.7). We can go back to Eq. (9.12) and integrate this over the whole heat exchanger as

$$(9.20) \quad \int_{\Delta T_{x=0}}^{\Delta T_{x=L}} d(\Delta T) = -C_0 \int_0^q dq',$$

from which we find that the total heat transfer q can be expressed as

$$\Delta T_{x=L} - \Delta T_{x=0} = -C_0 q.$$

Solving for C_0 , we obtain

$$C_0 = \frac{\Delta T_{x=0} - \Delta T_{x=L}}{q},$$

which can be substituted into Eq. (9.19)

$$(9.21) \quad \ln \left(\frac{\Delta T_{x=0}}{\Delta T_{x=L}} \right) = U_m A_t \frac{\Delta T_{x=0} - \Delta T_{x=L}}{q}.$$

Eq. (9.21) can be inverted as

$$(9.22) \quad q = U_m A_t \frac{\Delta T_{x=0} - \Delta T_{x=L}}{\ln(\Delta T_{x=0}/\Delta T_{x=L})}$$

Therefore, we see based on our original assumption in Eq. (9.7) about the form of the solution that

$$(9.23) \quad \Delta T_m = \frac{\Delta T_{x=0} - \Delta T_{x=L}}{\ln(\Delta T_{x=0}/\Delta T_{x=L})}.$$

This is the log mean temperature difference.

Now, what happens if $\Delta T_{x=0} = \Delta T_{x=L}$? It appears that we get a zero over zero situation, i.e. the numerator is 0 and the denominator is $\ln 1 = 0$. Mathematically, this is of course undefined and must be resolved. We can apply L'Hospital's rule, which says essentially that both numerator and denominator are differentiated until the limit of both can be meaningfully evaluated. In essence, we want to evaluate

$$\lim_{\Delta T_0 \rightarrow \Delta T_L} \frac{\Delta T_0 - \Delta T_L}{\ln(\Delta T_0/\Delta T_L)},$$

where we have shortened the notation for the temperatures. For the numerator, we evaluate

$$\frac{d}{d\Delta T_0} [\Delta T_0 - \Delta T_L] = 1 - 0.$$

Recall that derivatives of the natural log function are defined as

$$\frac{d}{dx} [\ln f(x)] = \frac{1}{f(x)} \frac{df}{dx},$$

so that for the denominator $f(\Delta T_0) = \Delta T_0/\Delta T_L$ we evaluate

$$\frac{d}{d\Delta T_0} \left[\ln \left(\frac{\Delta T_0}{\Delta T_L} \right) \right] = \frac{\Delta T_L}{\Delta T_0} \frac{1}{\Delta T_L} = \frac{1}{\Delta T_0}.$$

Reformulating the ratio, we now have

$$\lim_{\Delta T_0 \rightarrow \Delta T_L} \frac{1 - 0}{1/\Delta T_0} = \Delta T_0,$$

so in fact, the limit is ΔT_0 . Therefore, in the limiting case of $\Delta T_0 = \Delta T_L$, the log mean temperature difference $\Delta T_m = \Delta T_0 = \Delta T_L$.

We have mentioned in a qualitative sense that the counter-flow arrangement is somewhat "better" than parallel flow. Some quantitative light can be shed on this matter by estimating ΔT_m for each case. Assuming equal A_t and U_m , Eq. (9.7) indicates that heat transfer q depends upon ΔT_m . The expression given by Eq. (9.23) entails a subtle bookkeeping task of using the appropriate temperatures to compute $\Delta T_{x=0}$ and $\Delta T_{x=L}$. For example, the parallel flow situation in Fig. 9.7 implies

$$\Delta T_{x=0} = T_{h,i} - T_{c,i} \quad \text{and} \quad \Delta T_{x=L} = T_{h,o} - T_{c,o},$$

so that

$$\begin{aligned}\Delta T_{m, parallel} &= \frac{(T_{h,i} - T_{c,i}) - (T_{h,o} - T_{c,o})}{\ln([T_{h,i} - T_{c,i}] / [T_{h,o} - T_{c,o}])} \\ &= \frac{(T_{h,i} - T_{h,o}) - (T_{c,i} - T_{c,o})}{\ln([T_{h,i} - T_{c,i}] / [T_{h,o} - T_{c,o}])} \\ &= \frac{\Delta T_{hot side} + \Delta T_{cold side}}{\ln(\text{large number}/\text{small number})}\end{aligned}$$

Here, subscript “*i*” refers to an inlet temperature, while subscript “*o*” refers to an outlet temperature. Although the numerator is large, the denominator is not terribly small, i.e. it is the log of a rather large number.

Were we to duplicate the LMTD analysis for the counter-flow arrangement, we would once again realize Eq. (9.23), except that the bookkeeping aspect dictates

$$\Delta T_{x=0} = T_{h,i} - T_{c,o} \quad \text{and} \quad \Delta T_{x=L} = T_{h,o} - T_{c,i},$$

which can also be deduced from Fig. 9.4. We can then estimate ΔT_m as

$$\begin{aligned}\Delta T_{m, counter-flow} &= \frac{(T_{h,i} - T_{c,o}) - (T_{h,o} - T_{c,i})}{\ln([T_{h,i} - T_{c,o}] / [T_{h,o} - T_{c,i}])} \\ &= \frac{(T_{h,i} - T_{h,o}) - (T_{c,o} - T_{c,i})}{\ln([T_{h,i} - T_{c,o}] / [T_{h,o} - T_{c,i}])} \\ &= \frac{\Delta T_{hot side} - \Delta T_{cold side}}{\ln(\text{a number which may be close to unity})}\end{aligned}$$

IED Ex. 11.1
pp 655

Here, the numerator can be fairly small, but the denominator is the log of an argument that can be close to unity. Therefore, the denominator is *very* small, giving a larger LMTD temperature difference for the same input and output values (Table 9.3).

TABLE 9.3. Representative LMTD temperatures for parallel and counter-flow heat exchangers (all temperatures in °C)

Type	$T_{h,i}$	$T_{h,o}$	$T_{c,i}$	$T_{c,o}$	ΔT_m
parallel	100	80	60	78	12.69
counter-flow	100	80	60	78	20.98

9.4. Correction Factors for Complex Configurations

The LMTD analysis we developed is valid for simple parallel flow and counter flow units, but not for more complex multipass units (e.g. Fig. 9.5), or cross flow units (Fig. 9.8). In the latter design, fluids can either be “un-mixed”, meaning that transverse motion in a stream is not allowed, or “mixed” meaning that a stream can mix laterally.

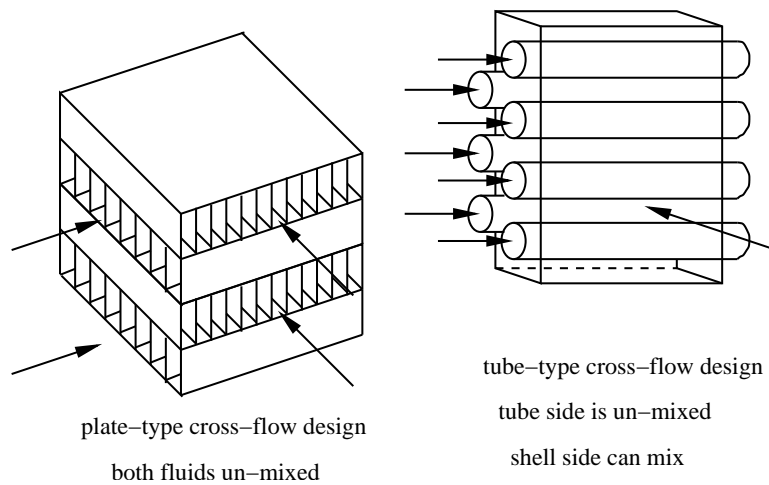


FIGURE 9.8. Cross flow heat exchanger designs for both fluids un-mixed (left) and one fluid mixed, one fluid un-mixed (right).

While such configurations have been analyzed, the results are usually considered too complicated for practical use. Instead, these results have been cast as correction factors to calculate appropriate LMTD temperatures. The form of the corrected LMTD temperature is

$$\Delta T_{m, corrected} = \Delta T_m F,$$

where F is the appropriate correction factor and ΔT_m is computed *on the basis of counterflow conditions!* References (e.g. Özişik, 1985; Incropera and Dewitt, 2002) typically present F graphically. Parameters in these graphical results depend by convention on the following nomenclature: t is the temperature on the tube side (hot or cold), while T is the temperature of the opposite (shell) stream. If the temperature change of one of the streams is negligible, as in a phase change, then $F = 1$. Generally, an exchanger with both sides being unmixed will give somewhat better performance than if one side is mixed because the streamwise temperature gradient is more conserved in the former.

*I&D Ex. 11.2
pp 657*

9.5. ϵ -NTU Analysis

For the LMTD method to be applicable, we must be able to compute ΔT_m , therefore we must know both the outlet and inlet temperatures for each stream. However, what if there is a situation in which the outlet temperatures are not known? For example, we may have a design problem where the inlet temperatures, fluid properties, and flow rates are known, but the outlet temperatures are not. Using LMTD analysis in this case would require a tedious iteration procedure to make sure that the heat transferred in the heat exchanger equals the heat energy gained by the cold side and lost by the hot side:

- Assume outlet temperatures and compute ΔT_m
- Determine q from Eq. (9.7), i.e. $q = A_t U_m \Delta T_m$
- Compute the outlet temperatures using q and the net temperature change of each stream
- Compare the outlet temperatures with those assumed in first step and adjust as necessary
- Repeat until suitable convergence is achieved

In this case, there is a better analysis method termed the “Effectiveness Method”, or commonly known as the ϵ -NTU method. First, we define the effectiveness of a heat exchanger as the actual heat transfer over the maximum possible heat transfer

$$(9.24) \quad \epsilon = \frac{q}{q_{max}} .$$

Before we delve into the ϵ -NTU method itself, let us also define *heat capacity rates* for the hot and cold side as

$$C_h = \dot{m}_h c_{p,h} \quad \text{and} \quad C_c = \dot{m}_c c_{p,c} .$$

According to what we have already concluded, q_{max} would be obtained using a counter-flow architecture having an infinite length. In this configuration, one of the fluids would experience the maximum possible temperature change

$$(9.25) \quad \Delta T_{max} = T_{h,i} - T_{c,i} ,$$

where $T_{h,i}$ and $T_{c,i}$ are again the inlet temperatures of the hot and cold streams, respectively. In this case the maximum heat transfer would be

$$(9.26) \quad q_{max} = C_{min} (T_{h,i} - T_{c,i}) ,$$

where

$$(9.27) \quad C_{min} = \min [C_h, C_c]$$

is the minimum of the two heat capacity rates. Can we prove that Eqs. (9.25) through (9.27) always represent the maximum temperature change of one of the streams?

There are only two possible cases: the cold side is the “minimum” side, or the hot side is the “minimum” side (Fig. 9.9). If the cold side is the minimum, then conservation of energy, written as

$$q = C_c \Delta T_c = C_h \Delta T_h ,$$

dictates that the cold side undergoes a larger change in temperature than the hot side, $\Delta T_c > \Delta T_h$, so that the cold side outlet temperature would be heated up to the value of the hot side inlet temperature, $T_{c,o} = T_{h,i}$. Therefore, the maximum energy transfer would be expressed as

$$q = C_c (T_{c,o} - T_{c,i}) = C_c (T_{h,i} - T_{c,i}) \quad \text{where} \quad C_c = C_{min} .$$

Conversely, if the hot side is the minimum, conservation of energy requires the hot side to experience the larger change in temperature, $\Delta T_h > \Delta T_c$,

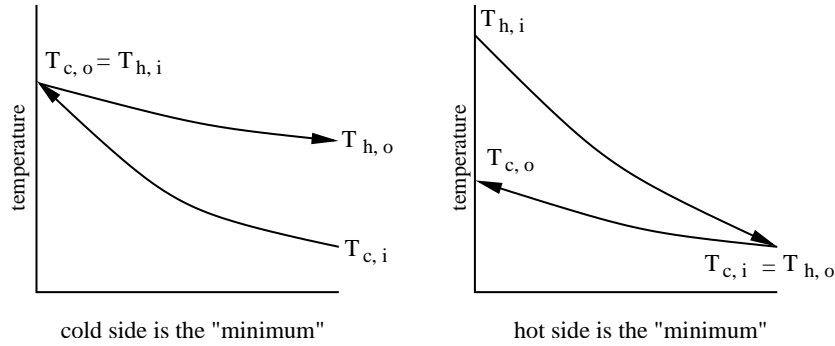


FIGURE 9.9. Diagrammatic representation of the concept of the "minimum" side for heat exchangers.

so that the hot side outlet temperature would be cooled to the value of the cold side inlet temperature, $T_{h,o} = T_{c,i}$. Now, maximum energy transfer would be written

$$q = C_h (T_{h,i} - T_{h,o}) = C_h (T_{h,i} - T_{c,i}) \quad \text{where} \quad C_h = C_{min}.$$

This proves the above concept.

Based on this exercise, it should be clear that that q_{max} is *not* based on the maximum $\dot{m}c_p$. If it were, then according to conservation of energy, the fluid with the minimum $\dot{m}c_p$ would undergo a temperature change greater than the maximum ΔT , which is not possible. According to Eq. (9.24), computing the heat transfer q depends on knowing ϵ . How do we determine this?

The parameter ϵ depends upon the heat exchanger design and flow arrangement. Here, we illustrate the general procedure for deriving ϵ assuming a parallel flow single pass arrangement^{9.5}. This derivation refers, once again to Fig. 9.7. From the definition in Eq. (9.24), we get

$$\epsilon = \frac{q}{C_{min} (T_{h,i} - T_{c,i})}.$$

Also, based upon the total heat transferred, we have

$$q = C_h (T_{h,i} - T_{h,o}) = C_c (T_{c,o} - T_{c,i}).$$

By definition, we then have

$$(9.28) \quad \epsilon = \frac{C_h (T_{h,i} - T_{h,o})}{C_{min} (T_{h,i} - T_{c,i})} = \frac{C_c (T_{c,o} - T_{c,i})}{C_{min} (T_{h,i} - T_{c,i})}.$$

^{9.5}It is important to realize that q_{max} is always based on the counter-flow arrangement, but the analysis is applied to whatever specific architecture is used, for example as shown here for the parallel flow arrangement.

Now, recall Eq. (9.19) that was used in the derivation of the LMTD analysis method, again for the parallel flow arrangement

$$\ln\left(\frac{\Delta T_{x=0}}{\Delta T_{x=L}}\right) = \ln\left(\frac{T_{h,i} - T_{c,i}}{T_{h,o} - T_{c,o}}\right) = U_m A_t C_0,$$

where we recall that

$$C_0 = \frac{1}{\dot{m}_h c_h} + \frac{1}{\dot{m}_c c_c} = \frac{1}{C_h} + \frac{1}{C_c}.$$

Taking the exponential, we can write

$$\frac{T_{h,i} - T_{c,i}}{T_{h,o} - T_{c,o}} = e^{U_m A_t C_0},$$

which inverts as

$$\frac{T_{h,o} - T_{c,o}}{T_{h,i} - T_{c,i}} = e^{-U_m A_t C_0}.$$

From Eq. (9.28), we can solve for $T_{h,o}$ to get

$$\begin{aligned} T_{h,o} &= T_{h,i} - \epsilon \frac{C_{\min}(T_{h,i} - T_{c,i})}{C_h} \\ &= T_{h,i} - \frac{C_c(T_{c,o} - T_{c,i})}{C_{\min}(T_{h,i} - T_{c,i})} \frac{C_{\min}(T_{h,i} - T_{c,i})}{C_h} \\ &= T_{h,i} - \frac{C_c(T_{c,o} - T_{c,i})}{C_h}, \end{aligned}$$

which can be used to eliminate $T_{h,o}$ from the exponential equation. This yields

$$\frac{T_{h,i} - \frac{C_c(T_{c,o} - T_{c,i})}{C_h} - T_{c,o}}{T_{h,i} - T_{c,i}} = e^{-U_m A_t C_0}.$$

Add and subtract $T_{c,i}$ to the numerator of the left hand side and we can show that the expression becomes

$$\begin{aligned} \frac{T_{h,i} - T_{c,i}}{T_{h,i} - T_{c,i}} + \frac{T_{c,i} - T_{c,o} - \frac{C_c(T_{c,o} - T_{c,i})}{C_h}}{T_{h,i} - T_{c,i}} &= e^{-U_m A_t C_0} \\ 1 + \frac{T_{c,i} \left(1 + \frac{C_c}{C_h}\right) - T_{c,o} \left(1 + \frac{C_c}{C_h}\right)}{T_{h,i} - T_{c,i}} &= \\ 1 + \left(1 + \frac{C_c}{C_h}\right) \frac{T_{c,i} - T_{c,o}}{T_{h,i} - T_{c,i}} &= e^{-U_m A_t C_0} \end{aligned}$$

Finally, this can be written as

$$\frac{T_{c,i} - T_{c,o}}{T_{h,i} - T_{c,i}} = \frac{e^{-U_m A_t C_0} - 1}{\left(1 + \frac{C_c}{C_h}\right)}$$

or

$$\frac{T_{c,o} - T_{c,i}}{T_{h,i} - T_{c,i}} = \frac{1 - e^{-U_m A_t C_0}}{\left(1 + \frac{C_c}{C_h}\right)}$$

Referring back to Eq. (9.28), we see that the above expression can be directly substituted to obtain ϵ , i.e.

$$\begin{aligned} \epsilon &= \frac{C_c}{C_{min}} \frac{1 - e^{-U_m A_t C_0}}{\left(1 + \frac{C_c}{C_h}\right)} \\ &= \frac{1 - e^{-U_m A_t C_0}}{\frac{C_{min}}{C_c} + \frac{C_{min}}{C_h}} \\ &= \frac{1 - \exp\left[-U_m A_t \left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right]}{C_{min} \left(\frac{1}{C_h} + \frac{1}{C_c}\right)} \\ &= \frac{1 - \exp\left[-\frac{U_m A_t}{C_{min}} C_{min} \left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right]}{C_{min} \left(\frac{1}{C_h} + \frac{1}{C_c}\right)}. \end{aligned}$$

Such expressions are normally written in a short-hand form, where we define the number of transfer units (NTU) as

$$NTU = \frac{U_m A_t}{C_{min}}$$

and the ratio of the minimum to maximum heat capacity rates as

$$C_1 = \frac{C_{min}}{C_{max}}.$$

It should be clear that, regardless of which side is the minimum side and which is the maximum,

$$C_{min} \left(\frac{1}{C_h} + \frac{1}{C_c}\right) = 1 + C_1,$$

which allows us to finally define ϵ for the parallel flow arrangement as

$$(9.29) \quad \epsilon = \frac{1 - e^{-NTU(1+C_1)}}{1 + C_1}.$$

This equation is quite significant in heat exchanger analysis, because one need not know the outlet temperatures. It suffices only to know the heat capacity rates^{9.6}, the area, and the overall convection coefficient to calculate ϵ . Moreover, because of the way ϵ is defined, the total heat transfer can be computed from a knowledge of only the inlet temperatures of both streams.

^{9.6}There is, however, one subtle issue: the specific heats may depend upon temperature. If we were to use the average temperature of a stream to approximate specific heat, we would need some *a priori* estimate of the exit temperature.

The procedure for analyzing other heat exchanger designs, e.g. counter-flow, multiple passes, etc., is similar but more complicated as compared to what we have shown here. If we simply reverse the design in our simple tube-type heat exchanger for counter-flow conditions, we find^{9.7}

$$(9.30) \quad \epsilon = \begin{cases} (1 - e^{-NTU(1-C_1)}) / (1 - C_1 e^{-NTU(1-C_1)}) & : C_1 < 1 \\ NTU / (NTU + 1) & : C_1 = 1 \end{cases}$$

Another notable configuration arises when there is a phase change (boiling or condensation), in which case ΔT of the phase change side essentially vanishes (Fig. 9.4). According to the fact that there is a finite thermal energy transfer, conservation of energy implies that $\dot{m}c_p$ can be taken as infinite. Therefore, the phase change side is always the “maximum” side and the heat capacity ratio can be taken as zero, i.e. $C_1 = 0$. Moreover, heat exchanger performance is independent of architecture and flow arrangement! The analysis gives

$$(9.31) \quad \epsilon = 1 - e^{-NTU} .$$

Reference texts (e.g. Incropera and Dewitt, 2002, Table 11.3, pp. 662–663) furnish ϵ -NTU relationships for a number of configurations and the inverted relations as well^{9.8}.

I&D Ex. 11.3
pp 665

^{9.7}The formula for this case in Incropera and Dewitt (2002, Table 11.3, pp. 662) is incorrect.

^{9.8}That is, NTU as a function of ϵ is provided for the cases that are difficult to invert.

CHAPTER 10

Introduction to Radiation

Up to now, we have considered our subject for the following general situation: heat transfer occurs when there is a temperature gradient in a participating medium. In conduction, the medium is not moving, while in convection, motion is the primary driver. Conversely, heat transfer by thermal radiation requires a temperature gradient but not participating medium. All bodies at a temperature above absolute zero emit thermal radiation. Consider for example a hot object in an evacuated container whose walls are cold. Conduction and convection are not possible, so heat is transferred strictly by radiation. Heat is transferred from the Sun to the Earth via radiation. However, the actual mechanism of radiation transfer is not yet fully understood. Maxwell's electromagnetic theory and Planck's quantum theory have both been used toward this goal.

If radiation is treated as a wave, the radiation from a body of temperature T is considered emitted at all wavelengths from $\lambda = 0$ to $\lambda = \infty$. For most thermal engineering applications, the range of interest is typically between $\lambda \approx 0.1 \mu m$ and $\lambda \approx 100 \mu m$ (Fig. 10.1). This portion of the spectrum

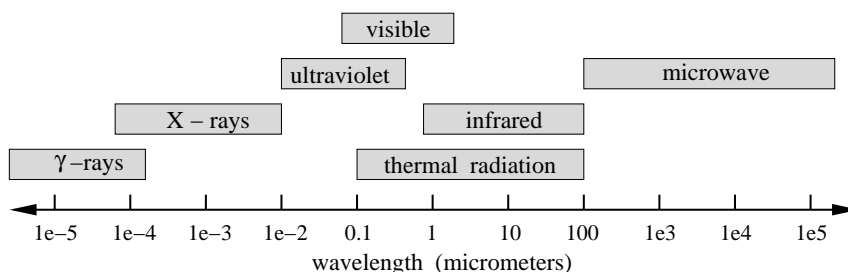


FIGURE 10.1. *The electromagnetic spectrum.*

is therefore usually referred to as the thermal radiation range. For example, solar radiation from the Sun ($T = 5760 K$) is in the range $\lambda \approx 0.1 \mu m$ to $\lambda \approx 3 \mu m$ (Özişik, 1985). Between $\lambda \approx 0.4 \mu m$ and $\lambda \approx 0.7 \mu m$ is visible light. For propagating radiation, the standard relationship between frequency ν and wavelength λ is

$$\lambda = \frac{c}{\nu},$$

where c is the speed of light in the medium^{10.1}.

Radiative heat transfer is quite a different phenomenon than conduction or convection. For example, in some materials, such as glass at certain temperatures, the radiation emitted from a body is a volumetric phenomenon. That is, it results as an integrated effect from the depths of the body all the way to its surface. Such bodies are called *semi-transparent*. Conversely, *opaque* bodies realize radiative effects mainly as a surface phenomenon. Radiation from an interior molecule is largely absorbed by its neighbors, so that any net emission results almost entirely from molecules near the surface of the body, i.e. within about $1\ \mu\text{m}$ (Özişik, 1985). We shall concentrate on this model for our studies.

There are two additional points to mention before we proceed. First is the *spectral* nature of radiation (Fig. 10.2). This means that the thermal

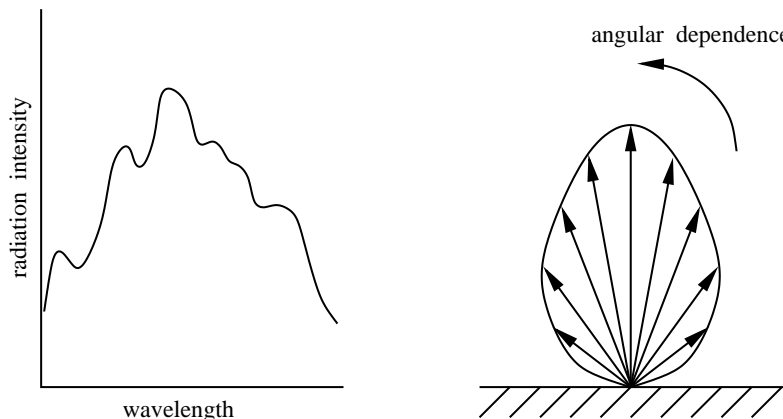


FIGURE 10.2. “Spectral” nature of radiation (left) implies a dependency upon wavelength, while the “directional” nature (right) implies dependence upon direction.

radiation emitted by a body encompasses a range of wavelengths, where the magnitude of the radiation varies with the wavelength. Second, radiation has a *directional* dependence: emissions are generally a function of direction. Because there are a number material properties associated with these phenomena, the “bookkeeping” aspect of radiation calculations is much more substantial than with other modes of heat transfer we have discussed thus far.

10.1. Solid Angle and Radiative Quantities

In order to quantify radiative heat transfer, we must first define a number of fundamental entities. First, we consider a new geometric concept called the *solid angle*, which will support the modeling of electromagnetic waves

^{10.1}For a vacuum, $c \approx 2.998 \times 10^8\ \text{m/s}$.

as rays that can be described according to straight-line optical principles. The solid angle can be thought of as the three-dimensional analog of the standard planar angle. For example, planar angle is defined as the unitless ratio^{10.2} of an arc length that is swept out by a radial arm to the arm's length (Fig. 10.3). This is naturally defined in the context of a circle by the

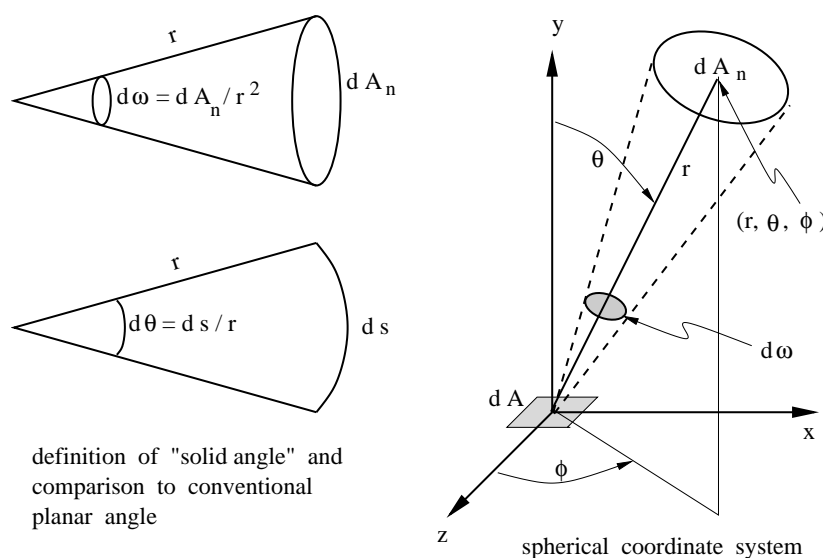


FIGURE 10.3. Definition of planar and solid angles and configuration of the solid angle in the spherical coordinate system.

differential expression^{10.3}

$$d\theta = \frac{ds}{r}.$$

Solid angle is instead defined in the context of a sphere as the ratio of a subtended area and the square of the radius^{10.4}. We express this as the differential

$$(10.1) \quad d\omega = \frac{dA_n}{r^2}.$$

We consider solid angle in the context of an observer anchored at the origin of a spherical coordinate system whose field of view encompasses a normally-oriented area dA_n a distance r away.

^{10.2}Although we attach a “unit” of *radians* to the planar angle, the quantity is indeed dimensionless. Angles expressed in the artificial system of *degrees* are not dimensionless.

^{10.3}This clear from the observation that we can integrate this equation to obtain the circumference as

$$\int_{\theta'=0}^{\theta'=2\pi} r d\theta' = \int_{s'=0}^{s'=s} ds, \quad \text{which gives} \quad s = r\theta' \Big|_{\theta'=0}^{\theta'=2\pi} = 2\pi r.$$

^{10.4}The “unit” of the solid angle is the *steradian*, (abbreviated as *sr*) which, like its planar counterpart, is actually dimensionless.

Actually, we must take this a step further and define solid angle in terms of differentials associated with the spherical coordinate system (Fig. 10.4). Consider emission from a differential element dA along radius r in a specific

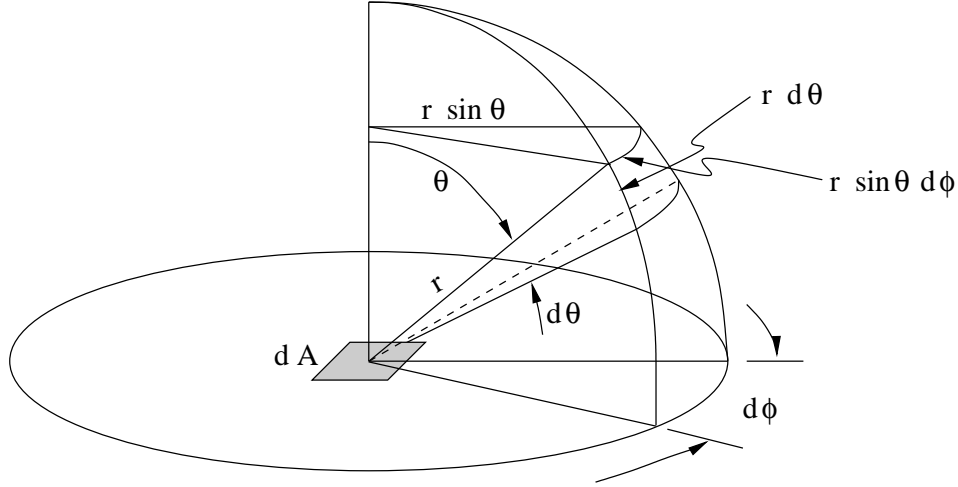


FIGURE 10.4. Relationship between solid angle and spherical coordinates.

direction defined by angles θ and ϕ . When viewed from dA , the emitted radiation passes through the differential element dA_n on the boundary of the hemisphere specified by radius r . By geometric arguments, we see

$$dA_n = (r d\phi \sin \theta) \times (r d\theta) = r^2 \sin \theta d\phi d\theta,$$

so that the solid angle relationship is^{10.5}

$$(10.2) \quad d\omega = \frac{dA_n}{r^2} = \frac{r^2 \sin \theta d\phi d\theta}{r^2} = \sin \theta d\phi d\theta.$$

Another geometric consideration in analyzing radiation emitted from dA is its projected area on dA_n . In other words, the question is how much of dA can an observer anchored on dA_n see? We should notice from Fig. 10.4 that when these areas are normal, i.e. $\theta = 0$, our observer will see all of dA , while at $\theta = \pi/2$, surface dA cannot be seen at all. A little trigonometry (Fig. 10.5) should convince us that

$$(10.3) \quad dA_n = dA \cos \theta.$$

The second concept is the fundamental quantity that describes radiative heat transfer, which is called the *radiation intensity*. Using this concept,

^{10.5}The solid angle for the entire hemisphere is

$$\omega = \int_{area} d\omega = \int_0^{2\pi} \int_0^{\pi/2} \sin \theta d\theta d\phi = \phi \Big|_0^{2\pi} (-\cos \theta) \Big|_0^{\pi/2} = 2\pi.$$

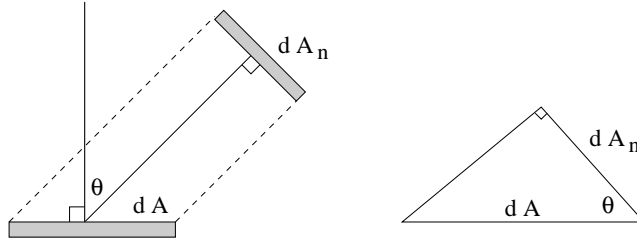


FIGURE 10.5. Area of the emitter projected onto the hemisphere boundary.

we will be able to quantify the spectral and directional properties of radiation, and integrate them into forms that are useful for engineering calculations. Again, the context is emission by differential surface dA which is intercepted by differential surface dA_n . Radiation intensity of the emitter, $I_{\lambda,e}$, is defined as the rate at which radiation dq is emitted in the (θ, ϕ) direction at wavelength λ per unit area of the emitter normal to this direction $dA_n = dA \cos \theta$, per unit solid angle about this direction $d\omega$, per unit wavelength interval $d\lambda$ about wavelength λ

$$(10.4) \quad I_{\lambda,e}(\lambda, \theta, \phi) = \frac{dq}{dA \cos \theta d\omega d\lambda}.$$

Intensity has units of $W / (m^2 sr \mu m)$. We can re-arrange Eq. (10.4) in terms of the rate at which radiation leaves the emitter dA and is intercepted by dA_n as

$$(10.5) \quad dq_\lambda = \frac{dq}{d\lambda} = I_{\lambda,e}(\lambda, \theta, \phi) dA \cos \theta d\omega,$$

where dq_λ is a shorthand notation and has units of $W/\mu m$. This expression allows us to compute radiation heat transfer as a function of radiation intensity, once it is known. Substituting our results for the solid angle from Eq. (10.2) and casting the heat transfer as a flux quantity with respect to dA , i.e. heat transfer per unit dA , we can write

$$(10.6) \quad dq''_\lambda = I_{\lambda,e}(\lambda, \theta, \phi) \cos \theta \sin \theta d\theta d\phi.$$

This provides a basis for integrating dq''_λ to find q''_λ , the flux associated with any particular finite solid angle. For example, the flux associated with the hemisphere around dA is

$$q''_\lambda = \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi) \cos \theta \sin \theta d\theta d\phi.$$

In other words, this expression gives the rate at which radiation at wavelength λ is emitted in all directions. In more direct terms, it is simply the rate at which radiation is emitted per unit area of the emitter. We define this as the *emissive power*, or, more exactly, the spectral hemispherical emissive

power^{10.6}

$$(10.7) \quad E_\lambda(\lambda) = \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi,$$

which has units of $W/(m^2 \mu m)$. The total hemispherical emissive power, i.e. the rate at which radiation is emitted in all directions at all wavelengths per unit area of the emitter, is obtained by integrating over λ

$$(10.8) \quad E = \int_0^\infty \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi \, d\lambda,$$

which has units of W/m^2 .

Although the directional distribution of surface radiation emission varies according to the properties of specific surfaces, there is an idealization which closely approximates conditions for many surfaces. This idealized form of radiation is called *diffuse radiation*, which simply indicates that there is no directional dependence. For example, for a diffuse emitter, the radiation intensity is independent of direction, so that $I_{\lambda,e}(\lambda, \theta, \phi) \rightarrow I_{\lambda,e}(\lambda)$. Since $I_{\lambda,e}(\lambda)$ is now constant with respect to directions, we see the emissive power in Eq. (10.7) can be cast as

$$\begin{aligned} E_\lambda(\lambda) &= \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda) \cos \theta \sin \theta \, d\theta \, d\phi \\ &= I_{\lambda,e}(\lambda) \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\phi \\ &= I_{\lambda,e}(\lambda) \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \\ &= I_{\lambda,e}(\lambda) \phi \Big|_0^{2\pi} \left(\frac{1}{2} \sin^2 \theta \right) \Big|_0^{\pi/2} \\ &= I_{\lambda,e}(\lambda) 2\pi \times \frac{1}{2} \\ (10.9) \quad E_\lambda(\lambda) &= \pi I_{\lambda,e}(\lambda) \quad (\text{diffuse emitter}) \end{aligned}$$

It can be similarly shown that $E = \pi I_e$, where again E is the total emissive power and I_e is the total radiation intensity.

IED Ex. 12.1
pp 707

Now let's look at some additional phenomena of interest in radiation heat transfer. First is the concept of *irradiation*, i.e. incoming or incident radiation. This is described by its own corresponding intensity, called the incident spectral intensity, $I_{\lambda,i}(\lambda, \theta, \phi)$, which, like the corresponding quantity for emitters, is a function of wavelength and direction and is given per unit area of the intercepting surface normal to this direction per unit solid angle, per unit $d\lambda$. The corresponding flux is the irradiation. Specifically, there is the spectral irradiation, $G_\lambda(\lambda)$, which is the rate at which radiation

^{10.6}We include the subscript λ in the notation for $E_\lambda(\lambda)$ to indicate that this is a per unit wavelength $d\lambda$ about λ quantity, synonymous with the way we have cast Eq. (10.5).

of wavelength λ is incident on a surface per unit area of surface per unit $d\lambda$ about λ from all directions. Also, there is the total irradiation, G , which is the rate at which radiation is incident per unit area from all directions at all wavelengths. This phenomenon obeys all the equations we discussed above if we substitute $E_\lambda \rightarrow G_\lambda$, $E \rightarrow G$, $I_{\lambda,e} \rightarrow I_{\lambda,i}$, etc.

IED Ex. 12.2
pp 710

Another phenomenon is that of *radiosity*, which accounts for all radiation leaving a surface, i.e. what is emitted directly plus whatever incident radiation is reflected. Therefore, this is generally not the same as the emissive power. We have all the corresponding quantities with their usual definitions: spectral radiosity intensity, $I_{\lambda,e+r}(\lambda, \theta, \phi)$, where $e + r$ indicates emission plus reflection, spectral radiosity, $J_\lambda(\lambda)$, and total radiosity, J . All corresponding relationships among these quantities are the same as those derived above.

10.2. Blackbody Radiation

When describing radiation with regard to real surfaces, it is useful to define the ideal case, called a *blackbody*, for comparison. It has three idealized properties

- A blackbody absorbs all incident radiation without reflecting, transmitting, or scattering it, regardless of wavelength and direction
- For a given temperature and wavelength, no surface can emit more energy than a blackbody — it is the ideal emitter
- While emitted radiation from a blackbody is a function of wavelength and temperature, it is independent of direction — a blackbody is a diffuse emitter

Thus, a blackbody is defined as a perfect absorber and emitter. The term “black” should be distinguished from its common usage regarding the appearance by visual observation of the blackness of a surface. The human eye can detect the color black only in the visual range of the radiation spectrum, but the visible range of light is only a small portion of the range of the thermal radiation spectrum (recall Fig. 10.1). For example, ice is non-black to the eye, however, for long wavelength thermal radiation, it is essentially black. Note that our definition of a true blackbody means that these properties must exist over the entire spectrum, i.e. $\lambda = 0 \rightarrow \infty$.

A blackbody, not surprisingly, is characterized by its own parameters, for example, the spectral blackbody radiation intensity, $I_{\lambda,b}(\lambda, T)$. Note that this does not depend on direction because of the diffuse nature of the radiation. This quantity was first determined by Max Planck for emission into a vacuum as

$$(10.10) \quad I_{\lambda,b}(\lambda, T) = \frac{2 h c^2}{\lambda^5 [e^{h c / (\lambda k T)} - 1]},$$

where $h = 6.6256 \times 10^{-34} J \cdot sec$ is the universal Planck constant, $k = 1.3805 \times 10^{-23} J/K$ is the Boltzmann constant, $c \approx 2.998 \times 10^8 m/sec$ is the

approximate speed of light in a vacuum, and T is the absolute temperature. Because the blackbody is a diffuse emitter, we can immediately derive the spectral blackbody emissive power from the relation $E_{\lambda,b}(\lambda, T) = \pi I_{\lambda,b}(\lambda, T)$ to get

$$E_{\lambda,b}(\lambda, T) = \frac{2 \pi h c^2}{\lambda^5 [e^{hc/(\lambda k T)} - 1]}.$$

Some texts (e.g. Özişik, 1985; Incropera and Dewitt, 2002) show this relation in terms of the so-called first and second radiation constants, $C_1 = 2\pi hc^2$ and $C_2 = hc/k$ giving^{10.7}

$$(10.11) \quad E_{\lambda,b}(\lambda, T) = \frac{C_1}{\lambda^5 [e^{C_2/(\lambda T)} - 1]}.$$

Eq. (10.11) is known as *Planck's Distribution* (Fig. 10.6) and has the fol-

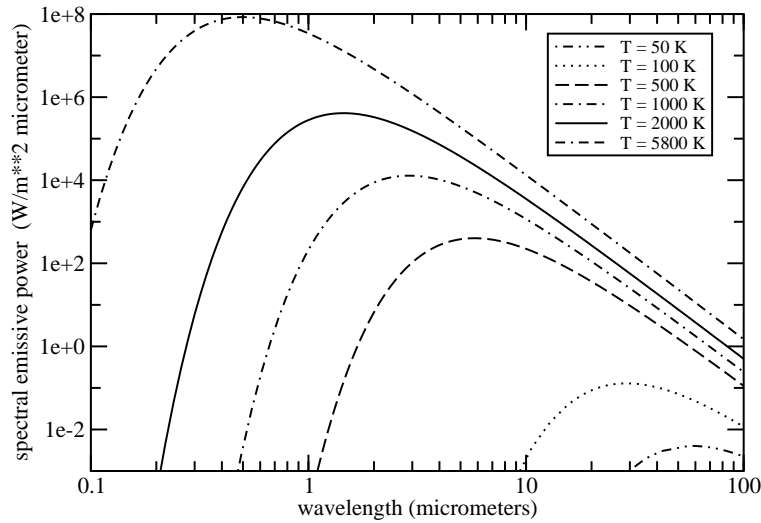


FIGURE 10.6. *Planck's Distribution of blackbody radiation.*

lowing notable properties

- Emitted radiation varies continuously with wavelength
- At any wavelength, the magnitude of emitted radiation increases with temperature
- The bulk of radiation is emitted at shorter wavelengths for higher temperatures
- The sun, approximated as a 5800 K blackbody, emits a significant fraction of radiation in the visible region of the spectrum.

^{10.7}These products yield $C_1 = 3.743 \times 10^8 \text{ W } \mu\text{m}^4/\text{m}^2$ and $C_2 = 14,387 \mu\text{m K}$.

The third point above has an interesting proof associated with it known as *Wien's Displacement Law*. If the Planck Distribution is differentiated with respect to λ and set equal to zero then solved, the result is

$$(10.12) \quad \lambda T \approx 2897.8 \mu m \cdot K .$$

This is shown in detail in Appendix C. The equation specifies the locus of peak points for $E_{\lambda,b}$. Because the product λT is constant, the maximum spectral emissive power drifts toward shorter wavelengths for higher temperatures.

What we are really interested in, i.e. what will enable us to solve problems, is the total blackbody emissive power. In principle, this can be derived by integrating the spectral blackbody emissive power given by Eq. (10.11) over all wavelengths

$$(10.13) \quad E_b(T) = \int_0^\infty \frac{C_1}{\lambda^5 (e^{C_2/(\lambda T)} - 1)} d\lambda .$$

Although this operation is not trivial^{10.8} it can be shown that

$$(10.14) \quad E_b(T) = \sigma T^4 ,$$

where $\sigma = 5.670 \times 10^{-8} W/(m^2 \cdot K^4)$ is the *Stefan-Boltzmann constant*, which depends upon the radiation constants C_1 and C_2 . This equation is known as the *Stefan-Boltzmann Law* and is important because it enables calculation of the amount of radiation emitted in all directions and over all wavelengths simply by knowing the blackbody temperature. Because the blackbody is diffuse, we can back out the total blackbody intensity as

$$I_b = \frac{E_b}{\pi} = \frac{\sigma T^4}{\pi} .$$

Often we need to know the fraction of radiative emission from a blackbody occurring over some finite wavelength range, e.g. $\lambda = 0 \rightarrow \lambda_1$. Rigorously, this can be computed by simply integrating Eq. (10.11) appropriately, e.g.

$$f_{0 \rightarrow \lambda_1} = \frac{\int_0^{\lambda_1} E_{\lambda,b} d\lambda}{\int_0^\infty E_{\lambda,b} d\lambda} = \frac{1}{\sigma T^4} \int_0^{\lambda_1} E_{\lambda,b} d\lambda$$

However, like the Stefan-Boltzmann Law, computation of this integral is not trivial. Introducing the Planck Distribution from above, we see that this gives

$$f_{0 \rightarrow \lambda_1} = \frac{1}{\sigma T^4} \int_0^{\lambda_1} \frac{C_1}{\lambda^5 (e^{C_2/(\lambda T)} - 1)} d\lambda .$$

Because this is so difficult to evaluate, we would like to leverage any analytical advantage that we can from this expression. We notice that $f_{0 \rightarrow \lambda_1}$ is a function of both wavelength and temperature, however, in this case, a clever reduction of two variables to one variable can be made.

^{10.8}This integral can be evaluated in terms of the Riemann zeta function.

Let us use the “Integration by Substitution” Theorem from Calculus (e.g. Gillman and McDowell, 1978, pp. 270), given by^{10.9}

$$(10.15) \quad \int_a^b f(u) \frac{du}{dx} dx = \int_{g(a)}^{g(b)} f(u) du ,$$

where $u = g(x)$. We want to write $f_{0 \rightarrow \lambda_1}$ in terms of the combined variable λT . Specifically, λ corresponds to x , and u maps to λT , therefore,

$$du = d(\lambda T) = \frac{d(\lambda T)}{d\lambda} d\lambda = T d\lambda ,$$

which leads to the substitution

$$d\lambda \rightarrow \frac{d(\lambda T)}{T} .$$

The upper limit is now $\lambda_1 \rightarrow \lambda_1 T$. Therefore, the equation can be rewritten as

$$\begin{aligned} f_{0 \rightarrow \lambda_1} &= \frac{1}{\sigma T^4} \int_0^{\lambda_1 T} \frac{C_1}{\lambda^5 (e^{C_2/(\lambda T)} - 1)} \frac{d(\lambda T)}{T} \\ &= \frac{C_1}{\sigma} \int_0^{\lambda_1 T} \frac{1}{(T \lambda)^5 (e^{C_2/(\lambda T)} - 1)} d(\lambda T) \end{aligned}$$

This expression can be recast in terms of $u = \lambda T$ as

$$(10.16) \quad f_{0 \rightarrow \lambda_1} = \frac{C_1}{\sigma} \int_0^{u_1} \frac{1}{u^5 (e^{C_2/u} - 1)} du .$$

The form and difficulty are similar to Eq. (10.13). Reference texts have tabulated f as a function of the combined variable λT . For example, Incropera and Dewitt (2002, Table 12.1, pp. 716) and Appendix C (Table C.1) give abbreviated summaries, while Dunkle (1954) and Siegel and Howell (2001) contain more comprehensive data.

^{10.9}This theorem can be proven in a straightforward fashion, for example, let F be the anti-derivative of function f and $u = g(x)$, so that

$$\begin{aligned} \int_a^b f(u) \frac{du}{dx} dx &= \int_a^b f(g(x)) g'(x) dx = F(g(x)) \Big|_a^b \\ &= F(g(b)) - F(g(a)) = F(u) \Big|_{g(a)}^{g(b)} = \int_{g(a)}^{g(b)} f(u) du , \end{aligned}$$

which proves Eq. (10.15).

Eq. (10.16) can be applied more generally for arbitrary ranges of radiation $\lambda = \lambda_1 \rightarrow \lambda_2$ according to

$$\begin{aligned}
 f_{\lambda_1 \rightarrow \lambda_2} &= \frac{\int_{\lambda_1}^{\lambda_2} E_{\lambda,b} d\lambda}{\sigma T^4} \\
 &= \frac{\int_0^{\lambda_2} E_{\lambda,b} d\lambda - \int_0^{\lambda_1} E_{\lambda,b} d\lambda}{\sigma T^4} \\
 &= \frac{\int_0^{\lambda_2} E_{\lambda,b} d\lambda}{\sigma T^4} - \frac{\int_0^{\lambda_1} E_{\lambda,b} d\lambda}{\sigma T^4} \\
 (10.17) \quad f_{\lambda_1 \rightarrow \lambda_2} &= f_{0 \rightarrow \lambda_2} - f_{0 \rightarrow \lambda_1}
 \end{aligned}$$

Note the special case for a range with a lower bound $\lambda = \lambda_2 \rightarrow \infty$ corresponds to

$$\begin{aligned}
 f_{\lambda_2 \rightarrow \infty} &= \frac{\int_{\lambda_2}^{\infty} E_{\lambda,b} d\lambda}{\sigma T^4} \\
 &= \frac{\int_0^{\infty} E_{\lambda,b} d\lambda - \int_0^{\lambda_2} E_{\lambda,b} d\lambda}{\sigma T^4} \\
 &= \frac{\int_0^{\infty} E_{\lambda,b} d\lambda}{\sigma T^4} - \frac{\int_0^{\lambda_2} E_{\lambda,b} d\lambda}{\sigma T^4} \\
 (10.18) \quad f_{\lambda_2 \rightarrow \infty} &= \frac{\sigma T^4}{\sigma T^4} - \frac{\int_0^{\lambda_2} E_{\lambda,b} d\lambda}{\sigma T^4} = 1 - f_{0 \rightarrow \lambda_2}
 \end{aligned}$$

IED Ex. 12.3
pp 717
IED Ex. 12.4
pp 719

10.3. Radiation Characteristics of Real Surfaces

10.3.1. Emissivity. Now that we have developed the concept of an ideal surface (such that we have a basis of comparison), let us now look at characteristics of real surfaces. In fact, we will specify properties of real surfaces via direct comparison to those of the blackbody. For example, recall that the blackbody, being the ideal emitter, emits the maximum radiation possible. Therefore, we can define the *emissivity* of a body as the ratio of radiation emitted by its surface to the radiation emitted by a blackbody at the same temperature. The spectrum of radiation emitted by a real surface is not necessarily the same as that specified by the Planck Distribution (Fig. 10.7) Also, the directional distribution may not qualify as diffuse. Therefore, emissivity can assume various forms according to whether one talks of emission at a given wavelength or emission in a given direction, or some overall integrated value.

First, we define a quantity called the spectral directional emissivity, $\varepsilon_{\lambda,\theta}(\lambda, \theta, \phi, T)$ of a surface at temperature T as the ratio of the intensity of the radiation emitted at wavelength λ in the (θ, ϕ) direction to the intensity of the radiation emitted by a blackbody for the same values of λ and T . This has the form

$$(10.19) \quad \varepsilon_{\lambda,\theta}(\lambda, \theta, \phi, T) = \frac{I_{\lambda,e}(\lambda, \theta, \phi, T)}{I_{\lambda,b}(\lambda, T)}.$$

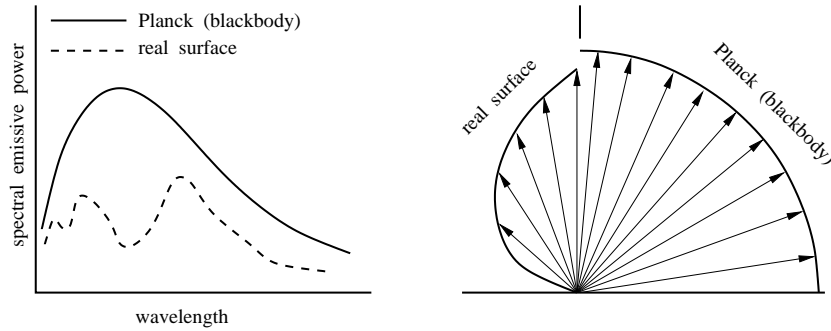


FIGURE 10.7. *Potential differences between the characteristics of real versus idealized (blackbody) surfaces for spectral (left) and directional (right) aspects of radiation emission.*

Note that the blackbody term in the denominator appears correctly as having no dependence upon direction since it is diffuse emitter. Likewise, the total directional emissivity, ε_θ , is an average over the wavelength spectrum and is defined as

$$(10.20) \quad \varepsilon_\theta(\theta, \phi, T) = \frac{I_e(\theta, \phi, T)}{I_b(T)}.$$

Note that these two quantities are actually defined in terms of radiation intensity because of directional dependence.

Conversely, we can also define an emissivity that has been averaged over all directions, called the spectral hemispherical emissivity

$$(10.21) \quad \varepsilon_\lambda(\lambda, T) = \frac{E_\lambda(\lambda, T)}{E_{\lambda,b}(\lambda, T)},$$

which is given in terms of emissive power. We recall from previous definitions that these quantities represent emissive powers that have been integrated over the hemispherical area of emission. How does this relate to the directional emissivity $\varepsilon_{\lambda,\theta}$? The spectral hemispherical emissivity can be written according to the definitions

$$(10.22) \quad \varepsilon_\lambda(\lambda, T) = \frac{\int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,e}(\lambda, \theta, \phi, T) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,b}(\lambda, T) \cos \theta \sin \theta \, d\theta \, d\phi}.$$

Using Eq. (10.19), substitute $I_{\lambda,e} = \varepsilon_{\lambda,\theta} I_{\lambda,b}$ into the top term, take note that $I_{\lambda,b}$ in both numerator and denominator do not depend upon direction (so they can be taken outside their integral signs), and the relation simplifies to

$$(10.23) \quad \varepsilon_\lambda(\lambda, T) = \frac{\int_0^{2\pi} \int_0^{\pi/2} \varepsilon_{\lambda,\theta}(\lambda, \theta, \phi, T) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\phi}.$$

If we make the further assumption that $\varepsilon_{\lambda,\theta}$ is independent of ϕ (a reasonable assumption for many surfaces), we can obtain (after evaluating the

denominator)

$$(10.24) \quad \varepsilon_\lambda(\lambda, T) = 2 \int_0^{\pi/2} \varepsilon_{\lambda,\theta}(\lambda, \theta, T) \cos \theta \sin \theta \, d\theta .$$

The total hemispherical emissivity, ε , accounts for all directions and all wavelengths and is defined as

$$(10.25) \quad \varepsilon(T) = \frac{E(T)}{E_b(T)} = \frac{E(T)}{\sigma T^4} .$$

Now, using our previously established definitions of $E = \int_0^\infty E_\lambda(\lambda) \, d\lambda$ and $E_\lambda(\lambda, T) = \varepsilon_\lambda(\lambda, T) E_{\lambda,b}(\lambda, T)$, we can see that

$$(10.26) \quad \varepsilon(T) = \frac{\int_0^\infty \varepsilon_\lambda(\lambda, T) E_{\lambda,b}(\lambda, T) \, d\lambda}{E_b(T)} .$$

If the emissivities of a surface are known, these definitions can be applied directly to calculate all emission characteristics.

We note (again according to definition) that the directional emissivity of a diffuse emitter is constant, i.e. independent of direction. For “real” surfaces, it may be said in general that

- emissivity of metallic surfaces is small
- oxide layers increase the emissivity of metallic surfaces
- emissivity of non-conductors is large
- emissivity of conductors increases with T , but may either decrease or increase for non-conductors

I&D Ex. 12.5
pp 724

I&D Ex. 12.6
pp 727

Moreover, emissivity can depend on other factors, for example local residual stresses or chemical interactions. Özişik (1985) and Incropera and Dewitt (2002) contain summary data, while more comprehensive data are available in reference texts (e.g. Wood et al., 1964).

10.3.2. Absorptivity. It follows that, like for emission, we can quantify the characteristics of irradiation in terms of surface properties. At the beginning of this chapter, we introduced the idea of a semi-transparent medium in which a portion of the irradiation is absorbed^{10.10}, a portion is reflected^{10.11}, and a portion is transmitted^{10.12}. For the most part, we will concentrate on situations where transmission vanishes, so that the problem is one of only absorption and reflection. Recall that such a material is called opaque.

A simple energy balance for opaque surfaces shows

$$(10.27) \quad G_\lambda = G_{\lambda,ref} + G_{\lambda,abs} + G_{\lambda,trans} ,$$

where G_λ is the irradiation, $G_{\lambda,ref}$ and $G_{\lambda,abs}$ are the components reflected and absorbed, respectively, and $G_{\lambda,trans}$ is the transmitted component, which

^{10.10}Manifested as heating at the surface.

^{10.11}Thrown back off away from the surface.

^{10.12}Allowed to pass through to lower layers of the material.

often vanishes. As we introduced various parameters that characterize emission, we likewise now introduce parameters that characterize absorption, transmission, and reflection^{10.13}. The spectral directional absorptivity determines the fraction of irradiation absorbed by a surface

$$(10.28) \quad \alpha_{\lambda,\theta}(\lambda, \theta, \phi) = \frac{I_{\lambda,i,abs}(\lambda, \theta, \phi)}{I_{\lambda,i}(\lambda, \theta, \phi)},$$

where these properties are not generally dependent upon surface temperature (as the corresponding parameters for emission were). The corresponding parameter after directional averaging is the spectral hemispherical absorptivity which is defined as

$$(10.29) \quad \alpha_{\lambda}(\lambda) = \frac{G_{\lambda,abs}(\lambda)}{G_{\lambda}(\lambda)},$$

which, from the definition

$$G_{\lambda} = \int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi$$

and Eq. (10.28), can be expressed directly as

$$(10.30) \quad \alpha_{\lambda}(\lambda) = \frac{\int_0^{2\pi} \int_0^{\pi/2} \alpha_{\lambda,\theta} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}.$$

Note that if the incident radiation is diffuse, $I_{\lambda,i}(\lambda, \theta, \phi) \rightarrow I_{\lambda,i}(\lambda)$ so this comes outside the integrals and cancels on top and bottom. Also, if $\alpha_{\lambda,\theta}$ is independent of ϕ we see that the expression becomes

$$(10.31) \quad \alpha_{\lambda}(\lambda) = 2 \int_0^{\pi/2} \alpha_{\lambda,\theta}(\lambda, \theta) \cos \theta \sin \theta \, d\theta.$$

in an analogous manner as Eq. (10.24) for the spectral emissivity.

The total hemispherical absorptivity is the integrated average over all directions and the entire wavelength spectrum and is defined simply as the fraction of the total irradiation absorbed by the surface

$$(10.32) \quad \alpha = \frac{G_{abs}}{G},$$

which, from the previous definition

$$G = \int_0^{\infty} G_{\lambda} \, d\lambda$$

and Eq. (10.29) can be expressed directly as

$$(10.33) \quad \alpha = \frac{\int_0^{\infty} \alpha_{\lambda}(\lambda) G_{\lambda}(\lambda) \, d\lambda}{\int_0^{\infty} G_{\lambda}(\lambda) \, d\lambda}.$$

^{10.13}Although again, we will largely stress absorption and reflection over transmission.

10.3.3. Reflectivity. Reflectivity determines the fraction of incident radiation reflected by a surface. The complicating factor here is that the phenomenon is inherently *bidirectional*. That is, it depends both upon the direction of the incident radiation and the direction of the reflected radiation. For now, we will avoid this problem by working with a form that represents the integrated average over the hemisphere associated with the reflected part of the radiation — thus, the directional dependence is simply associated with the incident radiation. Accordingly, the spectral directional reflectivity is defined as

$$(10.34) \quad \rho_{\lambda,\theta}(\lambda, \theta, \phi) = \frac{I_{\lambda,i,ref}(\lambda, \theta, \phi)}{I_{\lambda,i}(\lambda, \theta, \phi)},$$

the spectral hemispherical reflectivity is defined as

$$(10.35) \quad \rho_{\lambda}(\lambda) = \frac{G_{\lambda,ref}(\lambda)}{G_{\lambda}(\lambda)},$$

and the total hemispherical reflectivity is defined as

$$(10.36) \quad \rho = \frac{G_{ref}}{G}.$$

We can derive similar relations as we did for absorptivity, i.e.

$$(10.37) \quad \rho_{\lambda}(\lambda) = \frac{\int_0^{2\pi} \int_0^{\pi/2} \rho_{\lambda,\theta} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta d\theta d\phi}{\int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta d\theta d\phi}$$

and

$$(10.38) \quad \rho = \frac{\int_0^{\infty} \rho_{\lambda}(\lambda) G_{\lambda}(\lambda) d\lambda}{\int_0^{\infty} G_{\lambda}(\lambda) d\lambda}.$$

There are similar relationships for transmissivity τ .

10.3.4. Relationships. From the energy balance of irradiation we wrote in Eq. (10.27), it can be shown by dividing by the irradiation that^{10.14}

$$(10.39) \quad \rho_{\lambda} + \alpha_{\lambda} = 1$$

and, if properties are averaged over the entire wavelength spectrum

$$(10.40) \quad \rho + \alpha = 1.$$

I&D Ex. 12.7
pp 735

Therefore, knowledge of one property implies the other.

10.4. Kirchhoff's Law and Gray Surfaces

Energy conservation applied to irradiation gives rise to a relationship between absorptivity and reflectivity, shown in Eqs. (10.39) and (10.40). There is an additional set of relationships between absorptivity and emission, known as *Kirchhoff's Law*, which is derived as follows. Consider an isothermal blackbody enclosure at temperature T_s which contains a smaller object, also at temperature T_s (Fig. 10.8). The two bodies are in thermal

^{10.14}We assume here that the transmitted radiation vanishes, so that $G_{\lambda,trans} = 0$.

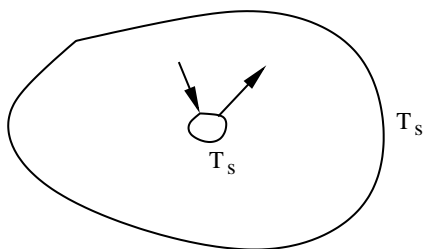


FIGURE 10.8. *Radiative transfer between two bodies at thermal equilibrium.*

equilibrium. The small object experiences a diffuse irradiation flux which is equal to the blackbody radiation flux emitted by the enclosure $G = E_b(T_s)$. The amount of radiation absorbed by the object is then obtained simply by multiplying by its absorptivity α and area A_{obj}

$$(10.41) \quad q_{abs} = \alpha A_{obj} G = \alpha A_{obj} E_b(T_s).$$

However, the body must emit radiation equal to the rate it is absorbed in order to satisfy thermal equilibrium, which implies $q_{emit} = q_{abs}$. Combining this observation with Eq. (10.41), we can write

$$(10.42) \quad q_{abs} = \alpha A_{obj} E_b(T_s) = q_{emit}.$$

From the emission standpoint, we recall that emissivity is defined as the ratio of the actual emissive power to the total emissive power, as quantified by Eq. (10.25). In particular, the emissivity of the small object is its actual emissive power divided by the emissive power of a blackbody operating at the same temperature. Converting these fluxes into absolute quantities based on the area of the small body, we can thus write

$$(10.43) \quad q_{emit} = \varepsilon A_{obj} E_b(T_s).$$

Eqs. (10.42) and (10.43) clearly imply

$$(10.44) \quad \varepsilon = \alpha$$

because they are otherwise equivalent. Note that in Eq. (10.42), the blackbody emissive power $E_b(T_s)$ applies to the walls of the enclosure, while in Eq. (10.43) the blackbody emissive power applies to our generic blackbody object operating at temperature T_s . Therefore, the restriction of thermal equilibrium is a primary requirement for the validity of Eq. (10.44). This equation is one form of Kirchhoff's Law.

By the same reasoning, both the spectral form of this law may be shown to be true

$$(10.45) \quad \varepsilon_\lambda = \alpha_\lambda,$$

and the spectral-directional version as well

$$(10.46) \quad \varepsilon_{\lambda,\theta} = \alpha_{\lambda,\theta}.$$

Eq. (10.46) is applicable to all cases, because $\varepsilon_{\lambda,\theta}$ and $\alpha_{\lambda,\theta}$ are independent of their respective overall spectral and directional distributions. What about the constraints on Eq. (10.45)? Are they as restrictive as those for Eq. (10.44), i.e. that the irradiation is from a blackbody operating at the same temperature?

Let us write Eq. (10.45) in terms of its respective definitions for ε_λ in Eq. (10.23) and α_λ in Eq. (10.30), which yields

$$\frac{\int_0^{2\pi} \int_0^{\pi/2} \varepsilon_{\lambda,\theta}(\lambda, \theta, \phi, T) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\phi} = \frac{\int_0^{2\pi} \int_0^{\pi/2} \alpha_{\lambda,\theta} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}{\int_0^{2\pi} \int_0^{\pi/2} I_{\lambda,i}(\lambda, \theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi}.$$

This relation, and thus Eq. (10.45), will be satisfied for either of the following two conditions

- If the irradiation $I_{\lambda,i}(\lambda, \theta, \phi)$ is diffuse then $I_{\lambda,i}(\lambda, \theta, \phi) \rightarrow I_{\lambda,i}(\lambda)$ so this comes outside the integrals and cancels, leaving the two sides equal since $\varepsilon_{\lambda,\theta}$ is always equal to $\alpha_{\lambda,\theta}$
- If the surface is diffuse, so that $\varepsilon_{\lambda,\theta}$ and $\alpha_{\lambda,\theta}$ are independent of direction, then they come outside the integrals, in which case the two sides are again equal

Are there any other conditions in which Eq. (10.44) will be satisfied? As with the spectral components, let us recast Eq. (10.44) in terms of its fundamental definitions for ε in Eq. (10.26) and α in Eq. (10.33). We find

$$\frac{\int_0^\infty \varepsilon_\lambda(\lambda, T) E_{\lambda,b}(\lambda, T) \, d\lambda}{E_b(T)} = \frac{\int_0^\infty \alpha_\lambda(\lambda) G_\lambda(\lambda) \, d\lambda}{G}.$$

Assuming $\varepsilon_\lambda = \alpha_\lambda$, this expression is satisfied for either of the following two conditions

- If the irradiation is the result of blackbody emission, then $G_\lambda(\lambda) = E_{\lambda,b}(\lambda, T)$ and $G = E_b(T)$, in which case the two sides are equal — this was our original condition
- If the surface is *gray*, meaning that both ε_λ and α_λ are independent of wavelength λ — they both come outside their respective integrals, so that the expression is once again satisfied

We see now that there are a number of forms of Kirchhoff's Law. For convenience, we summarize these forms and their restrictions in Table 10.1.

I&D Ex. 12.9

pp 742

I&D Ex. 12.10

pp 744

TABLE 10.1. Forms of Kirchhoff's Law

Form	Restrictions
$\varepsilon_{\lambda,\theta} = \alpha_{\lambda,\theta}$	none
$\varepsilon_{\lambda} = \alpha_{\lambda}$	$\varepsilon_{\lambda,\theta} = \alpha_{\lambda,\theta}$ and irradiation is diffuse <i>or</i> surface is diffuse
$\varepsilon = \alpha$	$\varepsilon_{\lambda} = \alpha_{\lambda}$ and thermal equilibrium and black irradiation <i>or</i> gray surface

CHAPTER 11

Radiation Exchange

In Chapter 10, we described some of the characteristics of thermal radiation. Now we extend this treatment to consider radiation exchange between two or more finite bodies.

11.1. The View Factor

Radiation exchange depends on surface geometries and their orientations, not to mention their radiative properties and temperatures. We will still invoke the idealization that any two surfaces are separated by a *non-participating medium*. That is, the medium has no affect upon the radiation exchange, for example, as with a vacuum. We will initially focus on the geometrical aspect by introducing the *view factor* concept^{11.1}. The view factor, F_{ij} , is defined simply as the fraction of radiation leaving surface i that is intercepted by surface j . To derive F_{ij} , we assume arbitrary surfaces A_i and A_j , which have respective differential areas dA_i and dA_j (Fig. 11.1). The

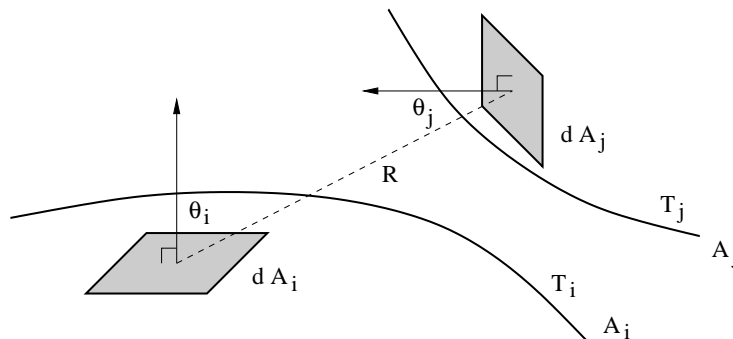


FIGURE 11.1. Calculation of the view factor.

differential areas under consideration are connected by a radius of length R , which forms polar angles θ_i and θ_j with the respective unit normal vectors. In general, R , θ_i , and θ_j are not constant as we consider the entirety of these two surfaces, i.e. the surfaces may have curvature, etc.

^{11.1}The terms “configuration factor” and “shape factor” used in some texts are synonymous.

From the form of Eq. (10.5) on pp. 120, we can deduce the rate at which radiation leaves dA_i and strikes dA_j as

$$dq_{i \rightarrow j} = I_i dA_i \cos \theta_i d\omega_{j,i},$$

where I_i is the intensity of the radiation leaving dA_i and $d\omega_{j,i}$ is the solid angle subtended by dA_j when viewed from dA_i . Now recall the definition of solid angle is dA_{normal}/R^2 , which means that, taking the projection of dA_j , gives

$$d\omega_{j,i} = \frac{dA_j \cos \theta_j}{R^2}.$$

Substituting, we find

$$dq_{i \rightarrow j} = I_i \frac{\cos \theta_i \cos \theta_j}{R^2} dA_i dA_j.$$

The total radiation coming from dA_i is that which is emitted plus that which is reflected, i.e. I_i should be written more exactly as $I_i \rightarrow I_{i,e+r}$, where the extra subscripts indicate emission and reflection.

Now we make an important assumption about radiation emanating from dA_i . We assume that this surface emits and reflects diffusely, so that the total radiation is given by the radiosity relationship $J_i = \pi I_{i,e+r}$. We can then cast $dq_{i \rightarrow j}$ as

$$(11.1) \quad dq_{i \rightarrow j} = J_i \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j.$$

We now make another critical assumption: J_i is taken to be uniform over surface i . Then, the total rate at which radiation leaves surface i and is intercepted by surface j can be found simply by integrating Eq. (11.1) over both surfaces, which gives

$$\begin{aligned} q_{i \rightarrow j} &= \int_{A_i} \int_{A_j} J_i \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j \\ &= J_i \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j. \end{aligned}$$

Uniformity in J_i allows us to take it outside the integral.

Now, from the definition of view factor, the radiation which leaves surface i and is intercepted by surface j is $q_{i \rightarrow j}$ and the total radiation that leaves surface i (emitted plus reflected) is $A_i J_i$, therefore, by definition, the view factor is

$$(11.2) \quad F_{ij} = \frac{\text{radiation from } i \text{ that hits } j}{\text{total radiation from } i} = \frac{q_{i \rightarrow j}}{A_i J_i}.$$

By inspection, we see that view factor can be calculated as

$$(11.3) \quad F_{ij} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j.$$

Eq. (11.3) is remarkable in the sense that, with the assumptions we have made, the view factor is dependent strictly on geometry. We do not require

IED Ex. 13.1
pp 798

any particular knowledge of the radiative quantities to calculate it. Again, the important assumptions we have made in deriving Eq. (11.3) are

- the medium does not participate,
- surface i emits and reflects diffusely,
- radiosity of surface i is uniform

By a similar derivation, the complementary view factor F_{ji} can be represented as

$$F_{ji} = \frac{\text{radiation from } j \text{ that hits } i}{\text{total radiation from } j} = \frac{q_{j \rightarrow i}}{A_j J_j},$$

from which we find

$$(11.4) \quad F_{ji} = \frac{1}{A_j} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi R^2} dA_i dA_j.$$

11.2. View Factor Algebra

Based upon the definitions in Eqs. (11.3) and (11.4), certain identities can be immediately deduced. For example, both F_{ij} and F_{ji} depend upon the same integral, which implies

$$(11.5) \quad F_{ij} A_i = F_{ji} A_j.$$

Eq. (11.5) is termed the *reciprocity rule* and is useful for finding one view factor when the other is known. This relation is the first result in a system of algebra we will construct in order to minimize the number of times that the view factor integral actually has to be computed for a specific radiation problem.

Let us now consider view factors from the standpoint of conservation of energy. Imagine an enclosure involving a particular surface, which may, in general, be concave, flat, or convex. This surface is denoted by cross-hatching in Fig. 11.2. Without loss of generality, we label this surface as

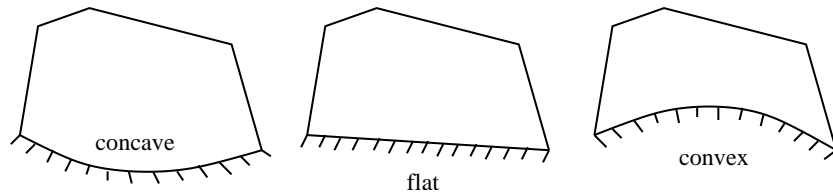


FIGURE 11.2. Enclosures for radiation exchange.

1, and the remaining surfaces in the enclosure $2, 3, \dots, N$. Note that one or more of the $2, 3, \dots, N$ surfaces may be “virtual” in the sense that they represent an unbounded area.

The total radiation emanating from surface 1 is $A_1 J_1$. In a proper accounting of how energy is conserved, all of this radiation must be intercepted by the other surfaces. In cases where surface 1 is concave, some of the emanating radiation will be self-intercepted because this surface can “see” itself.

In cases where the surface is flat or convex, the surface does not intercept any of its own emanating radiation. With these observations, we can write a simple conservation law for the radiation from surface 1 as

$$A_1 J_1 = q_{1 \rightarrow 1} + q_{1 \rightarrow 2} + q_{1 \rightarrow 3} + \cdots + q_{1 \rightarrow N}.$$

Dividing by $A_1 J_1$, we find

$$\frac{A_1 J_1}{A_1 J_1} = 1 = \frac{q_{1 \rightarrow 1}}{A_1 J_1} + \frac{q_{1 \rightarrow 2}}{A_1 J_1} + \frac{q_{1 \rightarrow 3}}{A_1 J_1} + \cdots + \frac{q_{1 \rightarrow N}}{A_1 J_1}.$$

Of course, the right hand side simply represents the sum of all the view factors, i.e.

$$1 = F_{11} + F_{12} + F_{13} + \cdots + F_{1N}.$$

Note according to the above discussion that $F_{11} = 0$ for flat and convex surfaces, but $F_{11} \neq 0$ for concave surfaces. We can write this in compact form for surface $i = 1$ as

$$(11.6) \quad \sum_{j=1}^N F_{ij} = 1,$$

which is called the *summation rule*.

Once again, the summation rule is a simple statement of conservation of radiative energy. We can write such an equation for each of the N surfaces in the enclosure, i.e. $1 \leq i \leq N$ equations. For any of these equations, the term F_{ii} represents self-interception of radiation for that particular surface. Generalizing the above observation, $F_{ii} = 0$ for flat and convex surfaces, but $F_{ii} \neq 0$ for concave surfaces.

Based on the observation in Eq. (11.6) that there are N view factors associated with each surface and there are N surfaces, the total number of view factors that characterize a radiation problem is N^2 , which can be expressed as the matrix

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots & F_{1N} \\ F_{21} & F_{22} & F_{23} & \cdots & F_{2N} \\ F_{31} & F_{32} & F_{33} & \cdots & F_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{N1} & F_{N2} & F_{N3} & \cdots & F_{NN} \end{bmatrix}.$$

Each of these view factors is defined by the view factor integral, as given by Eq. (11.3). In general, the view factor integral is not trivial to evaluate. However, Eqs. (11.5) and (11.6) clearly imply that not all of the factors has to be calculated directly.

Using the summation rule, N view factors can be obtained from the N equations derived from the summation rule applied to each surface of the enclosure. Also, $N(N-1)/2$ of the factors can be obtained by $N(N-1)/2$ applications of the reciprocity relation. This number is derived as follows: as

applied to a given surface there are $N - 1$ reciprocity relations^{11.2}, and this can be applied N times, thus giving $N(N - 1)$ relationships. However, only half of these are independent, giving $N(N - 1)/2$ relations^{11.3}. Therefore, only $N^2 - N - N(N - 1)/2 = N(N - 1)/2$ view factors actually have to be calculated directly. For example, for a 3-surface enclosure there are $3^2 = 9$ view factors, only $3 \cdot 2/2 = 3$ of which have to be computed directly. The other 6 factors can be calculated using the algebraic relationships in Eqs. (11.5) and (11.6).

EXAMPLE 11.1:

Consider a simple 2-surface enclosure of a sphere within a sphere (Fig. 11.3). The smaller sphere has an outer surface area of A_1 , while the

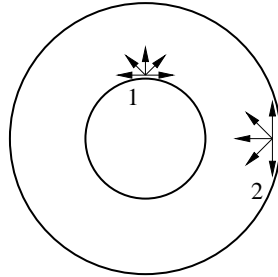


FIGURE 11.3. Two concentric spheres exchanging radiation.

larger has an inner surface area of A_2 . Determine the view factors characterizing this problem.

There are $N^2 = 2^2 = 4$ relevant view factors, only $N(N - 1)/2 = 2 \cdot 1/2 = 1$ of which has to actually be calculated directly, i.e. by evaluating the view factor integral. *The trick is to intelligently pick the view factor that must be calculated!* For example, all radiation leaving the inner sphere is intercepted by the outer sphere, therefore by inspection, we conclude $F_{12} = 1$. In this case, we skipped having to evaluate Eq. (11.3).

The rest of the problem is trivial. From reciprocity, we see

$$F_{21} = \frac{A_1}{A_2} F_{12} = \frac{A_1}{A_2}.$$

From summation, we see

$$F_{11} + F_{12} = 1 \quad \rightarrow \quad F_{11} = 0.$$

^{11.2}That is, there is a reciprocity relationship to every other surface besides the one in question. There is no meaningful reciprocity relationship between a surface and itself.

^{11.3}An even more direct proof is to realize that this problem is also quantified by the number of combinations of N entities taken 2 at a time. From combinatorics, we find

$$C_{N,2} = \frac{N!}{(N-2)!2!} = \frac{N \times (N-1) \times (N-2)!}{(N-2)! \times 2 \times 1} = \frac{N(N-1)}{2}.$$

Of course, this also could have been concluded by inspection, since the inner sphere is completely convex. This would have been the other easy starting point for the problem. Also from summation, we see

$$F_{21} + F_{22} = 1 \quad \rightarrow \quad F_{22} = 1 - \frac{A_1}{A_2}.$$

For more complex geometries, view factors may have to be calculated using the double-integral definition. Solutions have been obtained for a variety of configurations (e.g. Incropera and Dewitt, 2002, §13.1.2). $\diamond\diamond\diamond$

Another important identity can be derived by considering the case where a surface is broken up into its component parts (Fig. 11.4). First, it is evi-

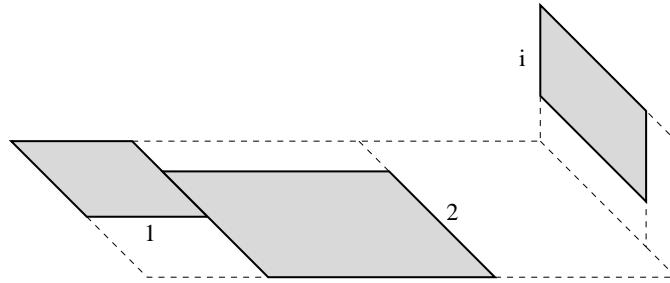


FIGURE 11.4. *Composite surfaces.*

dent that radiation reaching a composite surface is the sum of the radiation reaching each of its parts

$$(11.7) \quad F_{i(j)} = \sum_{k=1}^n F_{ik},$$

where the parentheses around j indicate that it is a composite surface made up of $1, 2, \dots, k, \dots, n$ components. The composite area is clearly

$$(11.8) \quad A_{(j)} = \sum_{k=1}^n A_k.$$

For example, in Fig. 11.4, we would write $F_{i(j)} = F_{i(1,2)} = F_{i1} + F_{i2}$ and $A_{(j)} = A_{(1,2)} = A_1 + A_2$. Turning this concept around via reciprocity, we can also determine a component-wise description of the surface that originates

the radiation. Multiplying Eq. (11.7) by A_i , we obtain

$$\begin{aligned}
 A_i F_{i(j)} &= A_i \sum_{k=1}^n F_{ik} \\
 &= A_i (F_{i1} + F_{i2} + \cdots + F_{in}) \\
 &= A_i F_{i1} + A_i F_{i2} + \cdots + A_i F_{in} \\
 &= A_1 F_{1i} + A_2 F_{2i} + \cdots + A_n F_{ni} \quad (\text{reciprocity}) \\
 &= \sum_{k=1}^n A_k F_{ki}.
 \end{aligned}$$

We can also write an “overall” reciprocity equation $A_i F_{i(j)} = A_{(j)} F_{(j)i}$, so that we finally conclude

$$(11.9) \quad F_{(j)i} = \frac{1}{A_{(j)}} \sum_{k=1}^n A_k F_{ki}.$$

Referring again to Fig. 11.4 as an example, we find

$$F_{(1,2)i} = \frac{A_1 F_{1i} + A_2 F_{2i}}{A_1 + A_2}.$$

IED Ex. 13.2
pp 799

Finally, other view factor relations may exist, depending upon the specific problem. For example, because we have already shown that the view factor integral in Eq. (11.3) depends only upon geometry, any geometric symmetry also implies symmetry in the corresponding view factors. Fig. 11.5 shows two examples for $N = 3$. In the triangular duct $F_{12} = F_{13}$. For

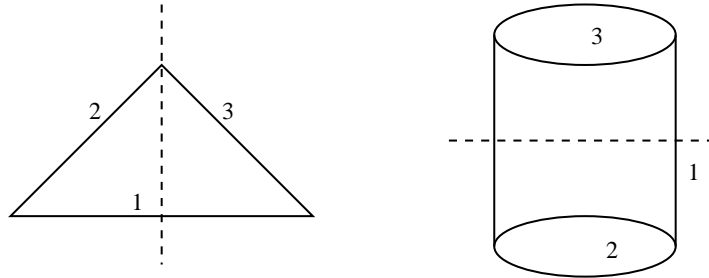


FIGURE 11.5. Radiation enclosures with $N = 3$ surfaces. Dotted lines indicate planes of symmetry.

the cylinder, surface 1 is taken as the entire inside surface, so that again $F_{12} = F_{13}$. Unlike Eqs. (11.5) and (11.6), such relations are problem specific and will not necessarily exist for every configuration.

In retrospect, we see that $N(N - 1)/2$ is actually the upper bound for the number of view factors that have to be calculated directly. Any extra relations that we can deduce, e.g. by inspection of flat or convex surfaces and by symmetry observations, decreases this number. Many problems are such that the view factor integral never actually has to be evaluated.

11.3. Blackbody Radiation Exchange

In the general problem, radiation departs from a surface via both emission and reflection. At the target, it may be reflected and absorbed. However, for blackbodies the situation is greatly simplified since there is no reflection. Energy only leaves a surface via emission and all incident radiation is absorbed at the target surface. For radiation exchange between two arbitrary blackbody surfaces, we can write

$$q_{i \rightarrow j} = J_i A_i F_{ij}$$

directly from the view factor definition in Eq. (11.2). Now, invoking the blackbody assumption, radiosity equals the emissive power for a blackbody, so that

$$q_{i \rightarrow j} = E_{b,i} A_i F_{ij}.$$

Similarly, we can write the radiation transfer in the opposite direction as

$$q_{j \rightarrow i} = E_{b,j} A_j F_{ji}.$$

The net exchange between the two surfaces can be written simply as the difference between these two quantities

$$\begin{aligned} q_{ij} &= q_{i \rightarrow j} - q_{j \rightarrow i} \\ &= E_{b,i} A_i F_{ij} - E_{b,j} A_j F_{ji} \\ &= E_{b,i} A_i F_{ij} - E_{b,j} A_i F_{ij} \quad (\text{reciprocity}) \\ &= A_i F_{ij} (E_{b,i} - E_{b,j}) \\ (11.10) \quad q_{ij} &= A_i F_{ij} \sigma (T_i^4 - T_j^4). \end{aligned}$$

Eq. (11.10) is the net rate at which radiation leaves surface i (is lost from surface i) as a result of interacting with surface j . Likewise, it is the net rate at which surface j gains energy.

This concept can easily be extended to an enclosure of N blackbody surfaces, all maintained at various temperatures. A similar treatment to the above yields

$$(11.11) \quad q_i = \sum_{j=1}^N A_i F_{ij} \sigma (T_i^4 - T_j^4) = \sigma A_i \sum_{j=1}^N F_{ij} (T_i^4 - T_j^4)$$

as the net transfer of radiation from surface i as a result of interacting with all of the other $N - 1$ surfaces of the enclosure.

11.4. Exchange Among Diffuse Gray Surfaces

Eq. (11.11) is interesting from a conceptual standpoint, but we have already seen that real surfaces often reflect radiation, i.e. $\rho > 0$ and $\alpha < 1$. The reality of this situation is that “packets” of radiation will interact with many surfaces consecutively, each which might absorb some portion and reflect the remainder. Two surfaces will therefore exchange energy at many

IED Ex. 13.3
pp 801

“levels”, as the radiation bounces back and forth. This bouncing effect complicates matters.

In this section, we will relax somewhat the assumption of a blackbody. Specifically, we will model enclosures having the following properties for each surface

- isothermal (uniform temperature)
- uniform radiosity and irradiation
- opaque ($\tau = 0$), diffuse, and gray ($\varepsilon = \alpha$)

and will then develop relations for the net rate at which radiation leaves a surface i , defined as q_i . In particular, we can write q_i according to conservation of energy at the radiative surface, either in terms of the difference between radiosity and irradiation, or equivalently, between emission and absorption (Fig. 11.6). We have

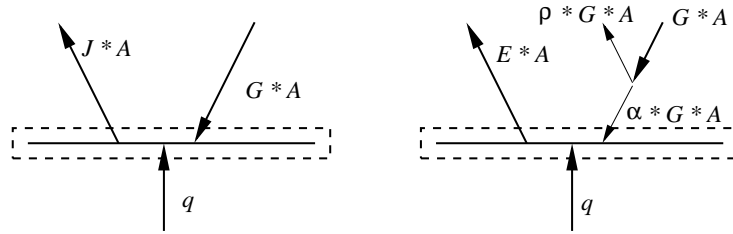


FIGURE 11.6. Conservation of energy in two equivalent forms, where q is the net loss from the surface.

$$\begin{aligned}
 (11.12) \quad q_i &= A_i (J_i - G_i) \\
 &= A_i ([E_i + \rho_i G_i] - G_i) \\
 &= A_i (E_i + G_i [\rho_i - 1]) \\
 &= A_i (E_i - G_i [1 - \rho_i])
 \end{aligned}$$

$$(11.13) \quad q_i = A_i (E_i - \alpha_i G_i).$$

Notice that the form containing radiosity implicitly includes absorption, while the form based on emission implicitly accounts for reflection. Let us focus on the radiosity $J_i = E_i + \rho_i G_i$. Using $E_i = \varepsilon_i E_{bi}$ and $\rho_i = 1 - \alpha_i = 1 - \varepsilon_i$, we can write radiosity as $J_i = \varepsilon_i E_{bi} + (1 - \varepsilon_i) G_i$. Solving for G_i , we find

$$G_i = \frac{J_i - \varepsilon_i E_{bi}}{1 - \varepsilon_i},$$

which, when substituted into q_i above, yields

$$\begin{aligned}
 q_i &= A_i \left(J_i - \frac{J_i - \varepsilon_i E_{bi}}{1 - \varepsilon_i} \right) \\
 &= A_i \left(J_i \frac{1 - \varepsilon_i}{1 - \varepsilon_i} - \frac{J_i - \varepsilon_i E_{bi}}{1 - \varepsilon_i} \right) \\
 &= A_i \left(\frac{J_i(1 - \varepsilon_i) - J_i + \varepsilon_i E_{bi}}{1 - \varepsilon_i} \right) \\
 &= A_i \left(\frac{\varepsilon_i E_{bi} - \varepsilon_i J_i}{1 - \varepsilon_i} \right) \\
 &= \frac{A_i \varepsilon_i (E_{bi} - J_i)}{1 - \varepsilon_i} \\
 (11.14) \quad q_i &= \frac{E_{bi} - J_i}{(1 - \varepsilon_i) / (A_i \varepsilon_i)}
 \end{aligned}$$

This expression is the net radiative energy from a surface. We readily notice the analogy to circuits, i.e. $E_{bi} - J_i$ represents driving potential, while q_i is heat flow (analogous to flow of electrical current) and $(1 - \varepsilon_i) / (A_i \varepsilon_i)$ is a radiative surface “resistance”. Fig. 11.7 illustrates this circuit analog. Conceptually, we can think of Eq. (11.14) as quantifying the amount by

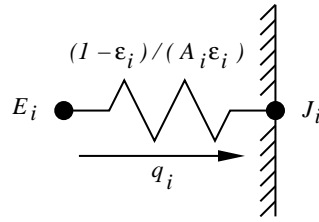


FIGURE 11.7. *Circuit analogy for radiative exchange at a surface as characterized by Eq. (11.14).*

which a real surface differs from the ideal blackbody radiator. In other words, if we write Eq. (11.14) as

$$q_i \frac{1 - \varepsilon_i}{A_i \varepsilon_i} = E_{bi} - J_i,$$

it is clear that $\varepsilon_i \rightarrow 1$ implies $J_i \rightarrow E_{bi}$. In other words, as the emissivity goes to unity, the radiosity approaches the blackbody emissive power — the surface is then equivalent to the ideal blackbody surface, for which $\varepsilon_i = \alpha_i = 1$ (perfect absorptivity) and $\rho_i = 0$ (no reflection). For $\varepsilon_i < 1$, the surface is not ideal, so that it is described by Eq. (11.14).

Unfortunately, the results expressed thus far are incomplete because we do not know the value of the radiosity J_i in Eq. (11.14). This must be calculated based on interaction with the other $N - 1$ surfaces of the problem. However, the radiosities for these surfaces then come into play as

well. Recalling the developments that led to the summation rule in Eq. (11.6) were based on the conservation of energy of the radiation leaving surface i . Let us now consider the conservation of energy for the irradiation of surface i , that is, the total irradiation is the sum of all net radiation transfers to surface i . We write this as

$$\begin{aligned}
 A_i G_i &= q_{1 \rightarrow i} + q_{2 \rightarrow i} + \cdots + q_{N \rightarrow i} \\
 &= A_1 J_1 F_{1i} + A_2 J_2 F_{2i} + \cdots + A_N J_N F_{Ni} \quad (\text{apply Eq. (11.2)}) \\
 &= J_1 A_1 F_{i1} + J_2 A_2 F_{i2} + \cdots + J_N A_N F_{Ni} \\
 &= J_1 A_i F_{i1} + J_2 A_i F_{i2} + \cdots + J_N A_i F_{iN} \quad (\text{reciprocity}) \\
 &= A_i (J_1 F_{i1} + J_2 F_{i2} + \cdots + J_N F_{iN}) .
 \end{aligned}$$

Canceling A_i from both sides, we can write the irradiation as

$$G_i = \sum_{j=1}^N J_j F_{ij} .$$

In turn, we can substitute this into Eq. (11.12) for the net rate of radiation leaving surface i , which yields

$$\begin{aligned}
 q_i &= A_i \left(J_i - \sum_{j=1}^N J_j F_{ij} \right) \\
 &= A_i \left(J_i \times 1 - \sum_{j=1}^N J_j F_{ij} \right) \quad (\text{multiply } J_i \text{ by } 1) \\
 &= A_i \left(J_i \sum_{j=1}^N F_{ij} - \sum_{j=1}^N J_j F_{ij} \right) \quad (\text{Eq. (11.6)}) \\
 &= A_i \left(\sum_{j=1}^N F_{ij} J_i - \sum_{j=1}^N J_j F_{ij} \right) \quad (\text{take } J_i \text{ into } \Sigma) \\
 (11.15) \quad q_i &= A_i \sum_{j=1}^N F_{ij} (J_i - J_j)
 \end{aligned}$$

Eq. (11.15) represents the net radiation leaving surface i written in terms of the interaction with radiosities of the other surfaces. We again notice a clear analogy to linear circuits where $J_i - J_j$ is the potential, q_i is flow, and $(A_i F_{ij})^{-1}$ is a resistance term^{11.4}. Unlike the single resistor implied by Eq. (11.14), this configuration represents a resistor for each surface interacting with the given surface i (Fig. 11.8). Surface i thus comprises the *node* for this circuit. Furthermore, it should be clear that in a given problem of N

^{11.4}This quantity is usually referred to as the *spatial resistance*, *geometric resistance*, or *shape resistance*.

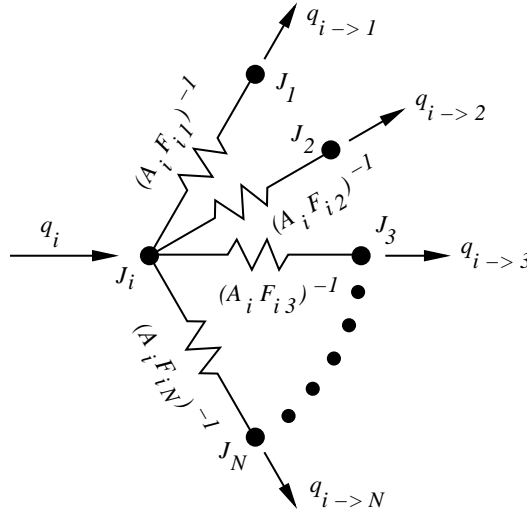


FIGURE 11.8. Circuit analogy for radiative exchange at a surface as characterized by Eq. (11.15).

surfaces, there will be N such circuits of the type shown in Fig. 11.8. That is, there will be such a circuit for each *node*, where the nodes go from 1 to N , each of which represents one surface of the problem.

11.5. The Graybody Matrix Problem

You have probably already deduced from the above discussion that the problem must ultimately be cast as a matrix equation. Like Eq. (11.14), there is insufficient information in Eq. (11.15) to solve the typical problem, because we generally do not know the values for q_i . However, we can readily set Eq. (11.14) equal to Eq. (11.15) to obtain

$$\frac{E_{bi} - J_i}{(1 - \varepsilon_i) / (A_i \varepsilon_i)} = A_i \sum_{j=1}^N F_{ij} (J_i - J_j) .$$

Or, canceling A_i , we can write more simply

$$(11.16) \quad \frac{E_{bi} - J_i}{(1 - \varepsilon_i) / \varepsilon_i} = \sum_{j=1}^N F_{ij} (J_i - J_j) .$$

Let us examine this equation closely. Eq. (11.16) is written for a specific surface i , so in an actual problem, there would be N such equations for $i : 1 \rightarrow N$. This equation is simply the conservation of energy for each node (surface) in the equation. In fact, it is really nothing more than *Kirchhoff's Circuit Law*, i.e. the sum of the currents (radiative transfers) for a node must be zero. This same concept appears in Fig. 11.8.

You have probably further deduced that the circuit analogy for this problem becomes too unwieldy for large N . For example, the circuit representation for $N = 3$ is shown in Fig. 11.9, but for higher values of N , such diagrams quickly become too difficult to reasonably draw. The matrix

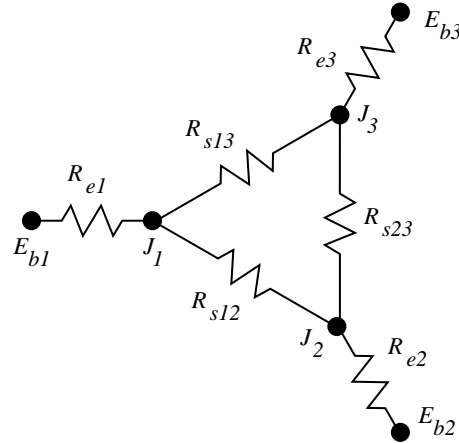


FIGURE 11.9. Circuit analogy for radiative exchange among 3 surfaces showing emissive powers, radiosity nodes, and all resistances. Resistors with subscript “e” are emissive resistors, while those with subscript “s” are shape resistors.

representation is rather more general. For the same $N = 3$ problem, we would write a form of Eq. (11.16) for each surface as

$$\begin{aligned} \frac{\varepsilon_1}{1 - \varepsilon_1}(E_{b1} - J_1) &= F_{11}(J_1 - J_1) + F_{12}(J_1 - J_2) + F_{13}(J_1 - J_3) \\ \frac{\varepsilon_2}{1 - \varepsilon_2}(E_{b2} - J_2) &= F_{21}(J_2 - J_1) + F_{22}(J_2 - J_2) + F_{23}(J_2 - J_3) \\ \frac{\varepsilon_3}{1 - \varepsilon_3}(E_{b3} - J_3) &= F_{31}(J_3 - J_1) + F_{32}(J_3 - J_2) + F_{33}(J_3 - J_3) \end{aligned}$$

where the unknowns are J_1 , J_2 , and J_3 . The emissive powers are considered known because they can be calculated from the Stefan–Boltzmann Law in Eq. (10.14) on pp. 124. Emissivities are also considered to be known, as are the view factors. These equations can be written in matrix form as

$$\begin{bmatrix} F_{12} + F_{13} + \bar{\varepsilon}_1 & -F_{12} & -F_{13} \\ -F_{21} & F_{21} + \bar{\varepsilon}_2 + F_{23} & -F_{23} \\ -F_{31} & -F_{32} & F_{31} + F_{32} + \bar{\varepsilon}_3 \end{bmatrix} \begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon}_1 E_{b1} \\ \bar{\varepsilon}_2 E_{b2} \\ \bar{\varepsilon}_3 E_{b3} \end{bmatrix}$$

where $\bar{\varepsilon}_1 = \varepsilon_1 / (1 - \varepsilon_1)$, $\bar{\varepsilon}_2 = \varepsilon_2 / (1 - \varepsilon_2)$, and $\bar{\varepsilon}_3 = \varepsilon_3 / (1 - \varepsilon_3)$ are shorthand notations. Of course, this particular system can be solved using any suitable matrix iteration or inversion procedure (Kreyszig, 1988). Configurations having $N > 3$ lead to commensurately larger matrix problems.

One configuration where the circuit analogy is preferred is the case of $N = 2$. This problem reduces to the simple series of resistors (Fig. 11.10).

I&D Ex. 13.4
pp 808

Here, we know the potential at the two termini from the Stefan–Boltzmann

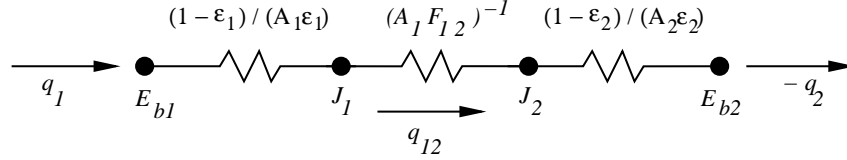


FIGURE 11.10. Series circuit for the $N = 2$ problem.

Law, and can thus write

$$(11.17) \quad q_{12} = \frac{\sigma (T_1^4 - T_2^4)}{(1 - \varepsilon_1) / (\varepsilon_1 A_1) + 1 / (A_1 F_{12}) + (1 - \varepsilon_2) / (\varepsilon_2 A_2)}.$$

Moreover, because there are only 2 surfaces, the net rate of transfer from surface 1 to surface 2, q_{12} , must be equal to the net rate of transfer from surface 1, q_1 , and thus the net rate of transfer to surface 2, $-q_2$.

11.6. Additional configurations

There are several interesting configurations related to radiation exchange. For example, if we examine the form of the emissive resistance in Eq. (11.14)

$$R = \frac{1 - \varepsilon}{A \varepsilon},$$

it is clear that R will be large if the emissivity is small. This provides a basis for using low emissivity materials as radiation shields to decrease the heat transfer between two bodies^{11.5}. That is, a low- ε surface can be placed between two bodies between which heat transfer is to be decreased. This would result in additional high-resistance elements in Fig. 11.10 with a commensurate decrease in q_{12} . For gray surfaces we have $\alpha + \rho = \varepsilon + \rho = 1$, so that low- ε implies that the surface is highly reflective of radiation.

Another interesting configuration is the *re-radiating surface*, which is characterized by zero net radiation transfer, $q_i = 0$. This is a reasonable model when the surface is well-insulated on the opposite side. From Eq. (11.14), we see that $q_i = 0$ implies $E_{bi} = J_i$, so that the radiosity at node i will be known. How does this effectively change the problem?

Let us carefully examine the example of a 3 surface enclosure, $N = 3$, where one of the surfaces is a re-radiating surface (Fig. 11.11). It is clear that $q_{13} = q_{23}$, because there is no radiative transfer through resistor R_{e3} . In other words, $q_3 = 0$ because this is a re-radiating surface. Also, $J_3 = E_{b3}$ is known. The circuit is thus reduced to a simpler parallel arrangement where one of the legs is R_{s12} and the other is $R_{s13} + R_{s23}$. The circuit can be solved

I&D Ex. 13.6
pp 815

^{11.5}For example, so-called low- ε glass is used in this capacity.

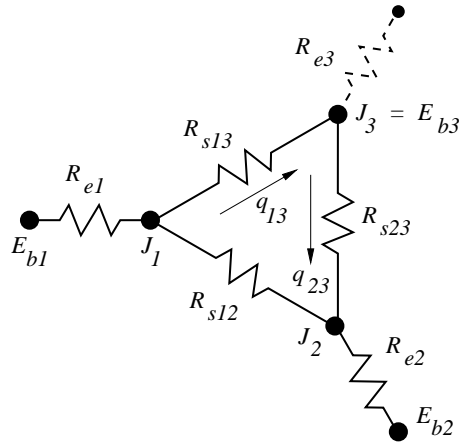


FIGURE 11.11. Circuit for $N = 3$ enclosure with surface 3 acting as a re-radiating surface.

by formulating the overall resistance between E_{b1} and E_{b2} . Using standard circuit analysis, the total resistance R_t is found to be

$$\begin{aligned} R_t &= R_{e1} + \frac{1}{\frac{1}{R_{s13} + R_{s23}} + \frac{1}{R_{s12}}} + R_{e2} \\ &= R_{e1} + \frac{R_{s12} (R_{s13} + R_{s23})}{R_{s12} + R_{s13} + R_{s23}} + R_{e2}, \end{aligned}$$

from which we can write the solution in the form

$$q_1 = -q_2 = \frac{E_{b1} - E_{b2}}{R_t}.$$

Notice that the problem is independent of resistor R_{e3} , whose form is

$$R_{e3} = \frac{1 - \varepsilon_3}{A_3 \varepsilon_3}.$$

In turn, this means that the problem is independent of the emissivity of surface 3. This observation is true in general, i.e. the emissivity of a re-radiating surface is irrelevant. Once q_1 is known, standard circuit theory can again be applied to find the potentials J_1 and J_2 . In a more general problem, i.e. $N > 3$, we can apply Eq. (11.15) in the standard fashion to formulate the matrix problem, except we simply set $q_i = 0$ for any re-radiating surface i .

Transient Conduction Example: The Homogeneous Cooling Problem

The concept of transient conduction was examined in Chapter 4. While certain idealizations were discussed, e.g. the lumped capacitance method, there were appreciable limitations upon their applicability. Here, we introduce a more generalized one-dimensional problem where spatial gradients do not vanish and the Fourier number is such that the one term approximation is not necessarily valid. We will focus on the case where boundary conditions are of the first kind (Dirichlet boundary conditions). There are a number of ways to approach this problem, however, we will use the well-known method of Separation of Variables.

Eq. (4.1) describes the physics of this problem

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2},$$

where T is only a function of x and t . Recall that this equation is a *linear* partial differential equation, so we suspect that we can treat specific problems strictly with theoretical methods^{A.1}. We take the problem domain to be a finite region in Cartesian space $0 \leq x \leq L$. Moreover, we assume a temporal duration of $t \geq 0$. To complete the description of a problem, that is to make sure it is well-posed, we will require both boundary conditions and initial conditions. We focus on *homogeneous* boundary conditions of the first kind, i.e. $T(0, t) = T(L, t) = 0$. It is assumed that specific non-homogeneous cases can be cast in homogeneous form using a suitable transformation. Initial conditions are given by a function $T(x, 0) = F(x)$.

A.1. Separation of Variables Method

The method begins with a conjecture that the solution can be written in the form

$$(A.1) \quad T(x, t) = \Psi(x) \Gamma(t).$$

That is, we assume that the physical problem is such that the contribution related to temporal response can be separated from the contribution related to spatial variation (Carrier and Pearson, 1976). For example, consider the imaginary case $T = (x^2 - 1)t$. This equation clearly represents a “separable” problem, where $\Psi = (x^2 - 1)$ and $\Gamma = t$. Conversely, the case $T = \sqrt{xt} +$

^{A.1}As opposed to non-linear problems which often cannot be solved analytically.

$\tanh(x^2\sqrt{t} + \sqrt{x/t})$ is not readily separable. We therefore suspect that even this method may be limited in the types of problems that may be solved. Fortunately, it is known to work for the type of one-dimensional configurations we are interested in here.

Under the conjecture of Eq. (A.1), partial derivatives have certain forms. Using the Chain Rule of Calculus, we see

$$(A.2) \quad \frac{\partial T}{\partial t} = \Gamma(t) \frac{\partial \Psi(x)}{\partial t} + \Psi(x) \frac{\partial \Gamma(t)}{\partial t} = \Psi(x) \Gamma'(t),$$

where the prime symbol denotes the derivative of a univariate function. Notice that the derivative of $\Psi(x)$ with respect to t vanishes because Ψ is only a function of x , not of t . Similarly, we can apply Chain Rule twice to find

$$(A.3) \quad \frac{\partial^2 T}{\partial x^2} = \Psi''(x) \Gamma(t).$$

Proceeding, we now substitute Eqs. (A.2) and (A.3) into the conduction equation to obtain

$$(A.4) \quad \frac{1}{\alpha} \frac{\Gamma'(t)}{\Gamma(t)} = \frac{\Psi''(x)}{\Psi(x)}.$$

We have now cast the problem in a *separated* form where the left hand side is only a function of time t and the right hand side is only a function of the spatial coordinate x . However, according to principle, Eq. (A.4) must be valid for all x and t in the problem domain. It follows that each side must be equal to a constant. Otherwise, either of x or t could be held fixed while the other could be varied such that Eq. (A.4) would be contradicted. Therefore,

$$(A.5) \quad \frac{1}{\alpha} \frac{\Gamma'(t)}{\Gamma(t)} = C_0$$

and

$$(A.6) \quad \frac{\Psi''(x)}{\Psi(x)} = C_0,$$

where C_0 is a constant. Positive values for C_0 lead to exponentially increasing behavior and imaginary values (involving $i = \sqrt{-1}$) lead to periodic behavior (Carrier and Pearson, 1976). These responses can be verified by substitution. We are instead interested in the case where C_0 is negative. This leads to behavior that decays exponentially in time, a phenomenon compatible with the type of boundary and initial conditions we will employ. Therefore, we define

$$(A.7) \quad C_0 = -\lambda^2,$$

where $\lambda > 0$.

We have now developed two individual ordinary differential equations from the single partial differential equation

$$(A.8) \quad \Gamma'(t) + \alpha\lambda^2\Gamma(t) = 0 \quad t \geq 0$$

and

$$(A.9) \quad \Psi''(x) + \lambda^2\Psi(x) = 0 \quad 0 \leq x \leq L.$$

These taken separately are each easier to solve than the original partial differential equation.

We must apply the same separation process to the boundary conditions. Using Eq. (A.1), the two boundary conditions can be written as

$$\Psi(0) \Gamma(t) = \Psi(L) \Gamma(t) = 0.$$

Now, $\Gamma(t)$ cannot vanish for arbitrary values of t , otherwise the whole solution would be trivial. Therefore we see that $\Psi(0)$ and $\Psi(L)$ must vanish instead. The boundary conditions are therefore

$$(A.10) \quad \Psi(0) = 0$$

and

$$(A.11) \quad \Psi(L) = 0.$$

We do not perform a similar reduction on the initial condition. This is handled differently as discussed below. The problem in (x, t) has now been completely separated into two simpler problems: a spatial problem in x and a temporal problem in t .

A.2. Solution Procedure

We are now faced with solving the individual problems in x and t . As mentioned above, we are focusing specifically on homogeneous boundary conditions of the first kind. Cases involving other combinations of boundary conditions are more complex and require a generalized treatment (Özişik, 1980).

The first step is to realize that Eq. (A.8) has a solution of the form

$$(A.12) \quad \Gamma(t) = e^{-\alpha\lambda^2 t}.$$

This can be readily verified by substitution. Next, we see that the trigonometric functions can satisfy Eq. (A.9) as

$$(A.13) \quad \Psi(x) = C_1 \sin \lambda x + C_2 \cos \lambda x,$$

where C_1 and C_2 are constants of integration. If we apply Eq. (A.10), we get

$$\Psi(0) = C_1 \sin 0 + C_2 \cos 0,$$

which implies that $C_2 = 0$. Now apply Eq. (A.11) to obtain

$$\Psi(L) = C_1 \sin \lambda L = 0.$$

It is clear that $C_1 \neq 0$, otherwise the entire solution would once again be trivial. Therefore, it must instead be the case that $\sin \lambda L = 0$, so that

$$(A.14) \quad \lambda L = n\pi,$$

where $n = 1, 2, 3, \dots$. That is, the problem is only satisfied for certain values of λ , called *eigenvalues*, i.e. $\lambda = n\pi/L$. Thus, this is an eigenvalue problem^{A.2}.

Note, we will now change the nomenclature to reflect the fact that there are many admissible eigen-related values, thus $\lambda \rightarrow \lambda_n$, $C_1 \rightarrow C_n$, and $\lambda_n = n\pi/L$. Also note that in this case “ $\sin \lambda_n x$ ” is called the eigenfunction. Each mode n yields an elementary solution to the problem. A general solution is therefore obtained by the linear superposition of all modes $n = 1 \rightarrow \infty$. This can be written as

$$(A.15) \quad T(x, t) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x) e^{-\alpha \lambda_n^2 t}.$$

Eq. (A.15) now satisfies both the governing equation and the boundary conditions. However, we have yet to determine mode coefficients C_n such that the initial conditions are satisfied.

A.3. Determining Mode Coefficients

Directly plugging in initial conditions at $t = 0$ yields

$$(A.16) \quad T(x, 0) = F(x) = \sum_{n=1}^{\infty} C_n \sin \lambda_n x,$$

however, this appears to be a dead end at first glance since there are an infinite number of unknown C_n values and only one equation. In actuality, we see that this problem boils down to one of expanding an arbitrary function $F(x)$ into a series. In this case, the series will be a Fourier “sin” series (Boyce and DiPrima, 1977). Let us review how such a series is constructed.

First, we make use of the fact that the eigenfunctions are *orthogonal*, that is they obey

$$\int_0^L \sin \lambda_n x \sin \lambda_m x dx = 0$$

for $m \neq n$ and

$$\int_0^L \sin \lambda_n x \sin \lambda_m x dx = N(\lambda_n)$$

for $m = n$, where $N(\lambda_n)$ is the *norm* and is defined simply as

$$N(\lambda_n) = \int_0^L \sin^2 \lambda_n x dx.$$

Özişik (1980) has an extensive discussion of orthogonality.

^{A.2}Özişik (1980) discusses eigenvalue problems in detail.

Next, we operate on Eq. (A.16) with $\int_0^L \sin \lambda_n x dx$ to obtain

$$(A.17) \quad \int_0^L F(x) \sin \lambda_n x dx = \int_0^L \sum_{m=1}^{\infty} C_m \sin \lambda_m x \sin \lambda_n x dx .$$

Note, the mode number symbol is arbitrary and we changed $n \rightarrow m$ in the summation to avoid ambiguity. We recall that the mode coefficients are constants rather than functions of x , so Eq. (A.17) can be simplified to

$$(A.18) \quad \int_0^L F(x) \sin \lambda_n x dx = \sum_{m=1}^{\infty} C_m \int_0^L \sin \lambda_m x \sin \lambda_n x dx .$$

The summation sign and coefficients have been taken outside of the integral. Now we make the observation that the right hand side vanishes except in the case where $m = n$ due to the orthogonality property. This can be better visualized if we write out the terms explicitly. We obtain

$$\begin{aligned} & C_1 \int_0^L \sin \lambda_1 x \sin \lambda_n x dx + \\ & C_2 \int_0^L \sin \lambda_2 x \sin \lambda_n x dx + \\ & C_3 \int_0^L \sin \lambda_3 x \sin \lambda_n x dx + \cdots + \\ & C_n \int_0^L \sin \lambda_n x \sin \lambda_n x dx + \\ & C_{n+1} \int_0^L \sin \lambda_{n+1} x \sin \lambda_n x dx + \\ & C_{n+2} \int_0^L \sin \lambda_{n+2} x \sin \lambda_n x dx + \cdots \end{aligned}$$

as the explicit representation of the series. Clearly, only mode C_n is non-zero, as governed by orthogonality. The rest of the modes are trivial and the summation itself vanishes. We can then simplify Eq. (A.18) to

$$(A.19) \quad \int_0^L F(x) \sin \lambda_n x dx = C_n \int_0^L \sin \lambda_n \sin \lambda_n x dx ,$$

which can be written using the norm as

$$(A.20) \quad \int_0^L F(x) \sin \lambda_n x dx = C_n N(\lambda_n) .$$

Solving Eq. (A.20), we find that the mode coefficients are given by

$$(A.21) \quad C_n = \frac{1}{N(\lambda_n)} \int_0^L F(x) \sin \lambda_n x dx .$$

It is left as an exercise to the reader to show that $N(\lambda_n)$ for this case is simply $L/2$.

Eq. (A.15) satisfies the governing equation and the boundary conditions while Eq. (A.21) determines coefficients such that the initial condition $F(x)$ is satisfied. If $F(x)$ is very complicated, evaluation of Eq. (A.21) may not be straightforward. Otherwise, it can usually be handled with the aid of standard integral tables. These two equations give the final solution to the problem specified above. Physically, this means that we know the temperature distribution $T(x, t)$ for the entire problem domain $0 \leq x \leq L$ and $t \geq 0$. We can therefore compute quantities of engineering interest such as heat flux at any position and any time.

A.4. Example: The Unit Initial Condition

As an example of finishing the problem to obtain the exact solution, let us consider the unit initial condition $F(x) = 1$, i.e. the temperature at any x at time $t = 0$ is unity. We must now evaluate the mode coefficients using Eq. (A.21) as^{A.3}

$$\begin{aligned}
 C_n &= \frac{1}{N(\lambda_n)} \int_0^L F(x) \sin \lambda_n x \, dx \\
 &= \frac{2}{L} \int_0^L 1 \sin \lambda_n x \, dx = \frac{2}{L} \int_0^L \sin \lambda_n x \, dx \\
 &= -\frac{2}{L \lambda_n} \cos \lambda_n x \Big|_0^L = -\frac{2L}{Ln\pi} \cos \frac{n\pi x}{L} \Big|_0^L \\
 &= -\frac{2}{n\pi} \left[\cos \frac{n\pi L}{L} - \cos 0 \right] \\
 &= \frac{2}{n\pi} [1 - \cos n\pi] \\
 &= \frac{2}{n\pi} [1 - (-1)^n]
 \end{aligned}$$

The last result is obtained by noting that n varies as $1, 2, 3, \dots$, for which the cosine of $n\pi$ must correspondingly alternate between negative and positive 1. In fact, we can construct a table of how the term in brackets varies with n (Table A.1). Clearly, all even-numbered modes drop out of the problem

TABLE A.1. Values of the alternating term: $[1 - (-1)^n]$

n	1	2	3	4	5	6	7	...
value	2	0	2	0	2	0	2	...

because their resulting mode coefficients are all 0. However, odd-numbered modes remain. We can simplify C_n further by writing it only for the non-trivial odd modes as

$$C_n = 2 \frac{2}{n\pi} = \frac{4}{n\pi} \quad \text{where } n = 1, 3, 5, 7, \dots$$

^{A.3}Recall that you've already showed that $N(\lambda_n) = L/2$.

Finally, we can substitute C_n into the general solution in Eq. (A.15) to obtain

$$T(x, t) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n\pi} \sin(\lambda_n x) e^{-\alpha\lambda_n^2 t}.$$

This expression can be used to evaluate temperature at any arbitrary (x, t) , e.g. by

- choose desired (x, t)
- set $T(x, t) = 0$
- set $n = 1$
- evaluate current mode n and add to running total of $T(x, t)$
- compare old value of $T(x, t)$ to newly calculated value
- stop if converged, i.e. if change of $T(x, t)$ is below user-specified convergence criterion, otherwise go to next mode

APPENDIX B

Laminar Forced Convection in a Pipe

This appendix shows the long-hand derivation of a number of equations relevant to laminar forced convection in fully developed pipe flow.

B.1. Volume Flow Rate

Eq. (7.4) gives the result for the volume flow rate in laminar pipe flow. Here, we formally show this starting from the definition of volume flow rate as the integral over the flow cross section

$$Q = \int_A \mathbf{V} \cdot \hat{n} dA = \int_0^{r_0} u 2\pi r dr$$

and the exact solution given by Eq. (7.3). Substituting this expression we find

$$\begin{aligned} Q &= - \int_0^{r_0} \frac{1}{4\mu} \left(\frac{dP}{dx} \right) r_0^2 \left[1 - \left(\frac{r}{r_0} \right)^2 \right] 2\pi r dr \\ &= - \frac{\pi r_0^2}{2\mu} \left(\frac{dP}{dx} \right) \int_0^{r_0} \left[1 - \left(\frac{r}{r_0} \right)^2 \right] r dr \\ &= - \frac{\pi r_0^2}{2\mu} \left(\frac{dP}{dx} \right) \int_0^{r_0} \left[r - \frac{r^3}{r_0^2} \right] dr \\ &= - \frac{\pi r_0^2}{2\mu} \left(\frac{dP}{dx} \right) \left[\frac{r^2}{2} - \frac{r^4}{4r_0^2} \right] \Big|_0^{r_0} \\ &= - \frac{\pi r_0^2}{2\mu} \left(\frac{dP}{dx} \right) \frac{r_0^2}{4} \\ &= - \frac{\pi r_0^4}{8\mu} \left(\frac{dP}{dx} \right), \end{aligned}$$

which confirms Eq. (7.4).

B.2. Integration of Mean Temperature Equation

Eq. (7.23) gives the result for the expression A in the case of constant wall heat flux. To show this, we must integrate Eq. (7.22), which has the form

$$\frac{2}{\bar{u} r_0^2} \int_0^{r_0} u \theta r dr = 1.$$

We substitute for u and θ to find

$$-\frac{2 A r_0^2}{\bar{u} r_0^2} \int_0^{r_0} 2 \bar{u} \left[1 - \left(\frac{r}{r_0} \right)^2 \right] \left[\frac{3}{16} + \frac{1}{16} \left(\frac{r}{r_0} \right)^4 - \frac{1}{4} \left(\frac{r}{r_0} \right)^2 \right] r dr = 1.$$

Writing out terms in long hand, we find

$$-4 A \int_0^{r_0} \left[\frac{3}{16} r - \frac{7}{16} \frac{r^3}{r_0^2} + \frac{5}{16} \frac{r^5}{r_0^4} - \frac{1}{16} \frac{r^7}{r_0^6} \right] dr = 1.$$

Carrying out term-by-term integration yields

$$\begin{aligned} -4 A \left(\frac{3}{2 \cdot 16} r^2 - \frac{7}{4 \cdot 16} \frac{r^4}{r_0^2} + \frac{5}{6 \cdot 16} \frac{r^6}{r_0^4} - \frac{1}{8 \cdot 16} \frac{r^8}{r_0^6} \right) \Big|_0^{r_0} &= 1 \\ -4 A \left(\frac{36 - 42 + 20 - 3}{24 \cdot 16} \right) r_0^2 &= \\ -4 A \left(\frac{11}{24 \cdot 16} \right) r_0^2 &= \\ -A \left(\frac{11}{6 \cdot 16} \right) r_0^2 &= \\ -\frac{11}{96} A r_0^2 &= 1 \end{aligned}$$

Solving, we find

$$A r_0^2 = -\frac{96}{11},$$

which confirms Eq. (7.23).

APPENDIX C

Blackbody Radiation

C.1. Wien's Displacement Law

Eq. (10.12) reports Wien's Displacement Law as

$$\lambda_{max} T = 2897.8 \mu m \cdot K .$$

Here, we prove this assertion starting from the spectral blackbody emissive power in Eq. (10.11)

$$E_{\lambda,b}(\lambda, T) = \frac{C_1}{\lambda^5 [e^{C_2/(\lambda T)} - 1]} .$$

We wish to determine the conditions for λ where $E_{\lambda,b}(\lambda, T)$ is a maximum. Differentiating with respect to λ and setting the result equal to zero, we find

$$\begin{aligned} \frac{dE_{\lambda,b}(\lambda, T)}{d\lambda} &= -5 C_1 \lambda^{-6} \left(e^{C_2/(\lambda T)} - 1 \right)^{-1} - \\ & C_1 \lambda^{-5} \left(e^{C_2/(\lambda T)} - 1 \right)^{-2} \left(-\frac{C_2}{T} \lambda^{-2} e^{C_2/(\lambda T)} \right) \\ &= -\frac{5}{\lambda} \cdot \frac{C_1}{\lambda^5 (e^{C_2/(\lambda T)} - 1)} + \\ & \frac{C_1}{\lambda^5 (e^{C_2/(\lambda T)} - 1)} \cdot \frac{1}{e^{C_2/(\lambda T)} - 1} \cdot \left(\frac{C_2}{T \lambda^2} e^{C_2/(\lambda T)} \right) \\ &= -\frac{5}{\lambda} \cdot E_{\lambda,b} + E_{\lambda,b} \cdot \frac{C_2}{T \lambda^2} \cdot \frac{e^{C_2/(\lambda T)}}{e^{C_2/(\lambda T)} - 1} \\ &= 0 , \end{aligned}$$

which can be recast as

$$-\frac{5}{\lambda} \cdot E_{\lambda,b} + E_{\lambda,b} \cdot \frac{C_2}{T \lambda^2} \cdot \frac{e^{C_2/(\lambda T)}}{e^{C_2/(\lambda T)} - 1} = 0 .$$

Factoring out $E_{\lambda,b}/\lambda$, we can now simplify this expression as

$$\frac{C_2}{T \lambda} \cdot \frac{e^{C_2/(\lambda T)}}{e^{C_2/(\lambda T)} - 1} = 5 .$$

Notice that the term $C_2/(\lambda T)$ appears as a unified variable. That is, we can let $x = C_2/(\lambda T)$, and solve the equation

$$x \frac{e^x}{e^x - 1} = 5 .$$

With a little algebra we find

$$x = 5 \frac{e^x - 1}{e^x} = 5 (1 - e^{-x}) ,$$

which is readily solved with a numerical procedure. For example, a few iterations of the Bisection Method (Kreyszig, 1988) shows

$$x = \frac{C_2}{\lambda T} \approx 4.965782 .$$

Substituting $C_2 = 14,390 \mu m \cdot K$ (Incropera and Dewitt, 2002) and solving, we find

$$\lambda T \approx \frac{C_2}{4.965782} = \frac{14390}{4.965782} = 2897.833 ,$$

which confirms Wien's Law in Eq. (10.12).

C.2. Radiation Functions

In §10.2, various blackbody radiation properties were discussed. Here, we give a summary of these properties in terms of the combined variable λT in units of $\mu m \cdot K$. Units of the column containing the radiation intensity are $(\mu m \cdot K \cdot sr)^{-1}$. This form, i.e. where temperature appears *to the minus fifth power*, is convenient for tabulation owing to the fact the wavelength appears as the negative fifth power in Planck's Distribution, e.g. Eqs. (10.10) and (10.11). See Dunkle (1954) and Siegel and Howell (2001) for more comprehensive data.

References show some variations due to differences in which integration is performed. For example, Incropera and Dewitt (2002, Table 12.1, pp. 716) shows $f_{0 \rightarrow \lambda} = 0.701046$ for $\lambda T = 5,600$, while Arpaci et al. (2000, Table 8.1, pp. 411) indicates this value is $f_{0 \rightarrow \lambda} = 0.70102$.

TABLE C.1. Blackbody radiation functions

λT	$I_{\lambda,b}/(\sigma T^5)$	$f_{0 \rightarrow \lambda}$	λT	$I_{\lambda,b}/(\sigma T^5)$	$f_{0 \rightarrow \lambda}$
200	3.7668e-28	0.0	5600	3.1624e-05	0.70102
400	4.9126e-14	0.0	5700	3.0396e-05	0.71077
500	2.1422e-11	0.0	5800	2.9217e-05	0.72013
600	1.0416e-09	0.0	5900	2.8086e-05	0.72914
700	1.4812e-08	0.0	6000	2.7000e-05	0.73779
800	9.9175e-08	0.000016	6100	2.5959e-05	0.74611
900	4.0592e-07	0.000087	6200	2.4961e-05	0.75411
1000	1.1855e-06	0.000321	6300	2.4005e-05	0.76181

continued on next page ...

λT	$I_{\lambda,b}/(\sigma T^5)$	$f_{0 \rightarrow \lambda}$	λT	$I_{\lambda,b}/(\sigma T^5)$	$f_{0 \rightarrow \lambda}$
1100	2.7224e-06	0.000911	6400	2.3088e-05	0.76920
1200	5.2404e-06	0.00213	6500	2.2210e-05	0.77632
1300	8.8324e-06	0.00432	6600	2.1369e-05	0.78317
1400	1.3442e-05	0.00779	6700	2.0563e-05	0.78976
1500	1.8889e-05	0.01285	6800	1.9792e-05	0.79610
1600	2.4913e-05	0.01972	6900	1.9053e-05	0.80230
1700	3.1227e-05	0.02853	7000	1.8345e-05	0.80808
1800	3.7554e-05	0.03934	7100	1.7667e-05	0.81373
1900	4.3654e-05	0.05211	7200	1.7018e-05	0.81918
2000	4.9337e-05	0.06673	7300	1.6396e-05	0.82443
2100	5.4466e-05	0.08305	7400	1.5800e-05	0.82949
2200	5.8955e-05	0.10089	7500	1.5229e-05	0.83437
2300	6.2760e-05	0.12003	7600	1.4682e-05	0.83910
2400	6.5873e-05	0.14026	7800	1.3656e-05	0.84801
2500	6.8312e-05	0.16136	8000	1.2713e-05	0.85625
2600	7.0113e-05	0.18312	8500	1.0672e-05	0.87457
2700	7.1325e-05	0.20536	9000	9.0103e-06	0.88999
2800	7.2005e-05	0.22789	9500	7.6496e-06	0.90304
2898	7.2212e-05	0.25011	10000	6.5296e-06	0.91416
2900	7.2212e-05	0.25056	10500	5.6024e-06	0.92367
3000	7.2005e-05	0.27323	11000	4.8308e-06	0.93185
3100	7.1441e-05	0.29578	11500	4.1852e-06	0.93892
3200	7.0576e-05	0.31810	12000	3.6421e-06	0.94505
3300	6.9458e-05	0.34011	12500	3.1831e-06	0.95401
3400	6.8132e-05	0.36173	13000	2.7932e-06	0.95509
3500	6.6640e-05	0.38291	13500	2.4605e-06	0.95921
3600	6.5017e-05	0.40360	14000	2.1753e-06	0.96285
3700	6.3295e-05	0.42376	14500	1.9299e-06	0.96607
3800	6.1501e-05	0.44337	15000	1.7178e-06	0.96893
3900	5.9657e-05	0.46241	15500	1.5338e-06	0.97149
4000	5.7785e-05	0.48087	16000	1.3736e-06	0.97377
4100	5.5901e-05	0.49873	16500	1.2336e-06	0.97581
4200	5.4019e-05	0.51600	17000	1.1109e-06	0.97765
4300	5.2151e-05	0.53268	18000	9.0777e-07	0.98081
4400	5.0306e-05	0.54878	19000	7.4879e-07	0.98341
4500	4.8492e-05	0.56430	20000	6.2298e-07	0.98555
4600	4.6716e-05	0.57926	25000	2.7633e-07	0.99217
4700	4.4982e-05	0.59367	30000	1.4039e-07	0.99529
4800	4.3293e-05	0.60754	35000	7.8621e-08	0.99695
4900	4.1654e-05	0.62089	40000	4.7364e-08	0.99792
5000	4.0065e-05	0.63373	45000	3.0200e-08	0.99852
5100	3.8528e-05	0.64608	50000	2.0150e-08	0.99890
5200	3.7043e-05	0.65795	55000	1.3952e-08	0.99917
5300	3.5610e-05	0.66936	75000	4.1837e-09	0.99971
5400	3.4230e-05	0.68034	100000	1.3568e-09	0.99991
5500	3.2902e-05	0.69088	∞	0	1.0

APPENDIX D

Document History

Version	Year	Remarks
1.0	1999	initial plain $\text{T}_{\text{E}}\text{X}$ implementation
2.0	2003	\LaTeX re-implementation using “amsbook” class; implementation of figures, tables, equation numbering, references, and index
2.1	2005	some minor corrections and additions; added page references where appropriate for better navigation

About the Author

Michael C. Wendl holds Bachelor's and Master's Degrees in Mechanical Engineering and a Doctorate in Applied Science and Engineering from Washington University. He is on the Faculties of Medicine (Genetics) and Engineering (Mechanical and Aerospace Engineering) at Washington University. Dr. Wendl's research activity primarily focuses on analytical work, especially in pure and applied mathematics, incompressible flow, conduction heat transfer, random DNA processing, and other areas of theoretical biology. His main teaching goal is to prepare and train students to think critically not only about science/engineering, but also broader issues that arise in life, e.g. decision-making, evaluating claims objectively, and logical problem solving.

Creative Commons Public License

Attribution–NoDerivs–NonCommercial 1.0

THE WORK (AS DEFINED BELOW) IS PROVIDED UNDER THE TERMS OF THIS CREATIVE COMMONS PUBLIC LICENSE (“CCPL” OR “LICENSE”). THE WORK IS PROTECTED BY COPYRIGHT AND/OR OTHER APPLICABLE LAW. ANY USE OF THE WORK OTHER THAN AS AUTHORIZED UNDER THIS LICENSE IS PROHIBITED.

BY EXERCISING ANY RIGHTS TO THE WORK PROVIDED HERE, YOU ACCEPT AND AGREE TO BE BOUND BY THE TERMS OF THIS LICENSE. THE LICENSOR GRANTS YOU THE RIGHTS CONTAINED HERE IN CONSIDERATION OF YOUR ACCEPTANCE OF SUCH TERMS AND CONDITIONS.

1. Definitions.

- a. “Collective Work” means a work, such as a periodical issue, anthology or encyclopedia, in which the Work in its entirety in unmodified form, along with a number of other contributions, constituting separate and independent works in themselves, are assembled into a collective whole. A work that constitutes a Collective Work will not be considered a Derivative Work (as defined below) for the purposes of this License.
- b. “Derivative Work” means a work based upon the Work or upon the Work and other pre-existing works, such as a translation, musical arrangement, dramatization, fictionalization, motion picture version, sound recording, art reproduction, abridgment, condensation, or any other form in which the Work may be recast, transformed, or adapted, except that a work that constitutes a Collective Work will not be considered a Derivative Work for the purpose of this License.
- c. “Licensor” means the individual or entity that offers the Work under the terms of this License.
- d. “Original Author” means the individual or entity who created the Work.
- e. “Work” means the copyrightable work of authorship offered under the terms of this License.
- f. “You” means an individual or entity exercising rights under this License who has not previously violated the terms of this License with respect to the Work, or who has received express permission from the Licensor to exercise rights under this License despite a previous violation.

2. Fair Use Rights. Nothing in this license is intended to reduce, limit, or restrict any rights arising from fair use, first sale or other limitations on the exclusive rights of the copyright owner under copyright law or other applicable laws.

3. License Grant. Subject to the terms and conditions of this License, Licensor hereby grants You a worldwide, royalty-free, non-exclusive, perpetual (for the duration of the applicable copyright) license to exercise the rights in the Work as stated below:

- a. to reproduce the Work, to incorporate the Work into one or more Collective Works, and to reproduce the Work as incorporated in the Collective Works;
- b. to distribute copies or phonorecords of, display publicly, perform publicly, and perform publicly by means of a digital audio transmission the Work including as incorporated in Collective Works;

The above rights may be exercised in all media and formats whether now known or hereafter devised. The above rights include the right to make such modifications as are technically necessary to exercise the rights in other media and formats. All rights not expressly granted by Licensor are hereby reserved.

4. Restrictions. The license granted in Section 3 above is expressly made subject to and limited by the following restrictions:

- a. You may distribute, publicly display, publicly perform, or publicly digitally perform the Work only under the terms of this License, and You must include a copy of, or the Uniform Resource Identifier for, this License with every copy or phonorecord of the Work You distribute, publicly display, publicly perform, or publicly digitally perform. You may not offer or impose any terms on the Work that alter or restrict the terms of this License or the recipients’ exercise of the rights granted hereunder. You may not sublicense the Work. You must keep intact all notices that refer to this License and to the disclaimer of warranties.

You may not distribute, publicly display, publicly perform, or publicly digitally perform the Work with any technological measures that control access or use of the Work in a manner inconsistent with the terms of this License Agreement. The above applies to the Work as incorporated in a Collective Work, but this does not require the Collective Work apart from the Work itself to be made subject to the terms of this License. If You create a Collective Work, upon notice from any Licensor You must, to the extent practicable, remove from the Collective Work any reference to such Licensor or the Original Author, as requested.

- b. You may not exercise any of the rights granted to You in Section 3 above in any manner that is primarily intended for or directed toward commercial advantage or private monetary compensation. The exchange of the Work for other copyrighted works by means of digital file-sharing or otherwise shall not be considered to be intended for or directed toward commercial advantage or private monetary compensation, provided there is no payment of any monetary compensation in connection with the exchange of copyrighted works.
- c. If you distribute, publicly display, publicly perform, or publicly digitally perform the Work or any Collective Works, You must keep intact all copyright notices for the Work and give the Original Author credit reasonable to the medium or means You are utilizing by conveying the name (or pseudonym if applicable) of the Original Author if supplied; the title of the Work if supplied. Such credit may be implemented in any reasonable manner; provided, however, that in the case of a Collective Work, at a minimum such credit will appear where any other comparable authorship credit appears and in a manner at least as prominent as such other comparable authorship credit.

5. Representations, Warranties and Disclaimer.

- a. By offering the Work for public release under this License, Licensor represents and warrants that, to the best of Licensor's knowledge after reasonable inquiry:
 - i. Licensor has secured all rights in the Work necessary to grant the license rights hereunder and to permit the lawful exercise of the rights granted hereunder without You having any obligation to pay any royalties, compulsory license fees, residuals or any other payments;
 - ii. The Work does not infringe the copyright, trademark, publicity rights, common law rights or any other right of any third party or constitute defamation, invasion of privacy or other tortious injury to any third party.
- b. EXCEPT AS EXPRESSLY STATED IN THIS LICENSE OR OTHERWISE AGREED IN WRITING OR REQUIRED BY APPLICABLE LAW, THE WORK IS LICENSED ON AN "AS IS" BASIS, WITHOUT WARRANTIES OF ANY KIND, EITHER EXPRESS OR IMPLIED INCLUDING, WITHOUT LIMITATION, ANY WARRANTIES REGARDING THE CONTENTS OR ACCURACY OF THE WORK.

6. Limitation on Liability. EXCEPT TO THE EXTENT REQUIRED BY APPLICABLE LAW, AND EXCEPT FOR DAMAGES ARISING FROM LIABILITY TO A THIRD PARTY RESULTING FROM BREACH OF THE WARRANTIES IN SECTION 5, IN NO EVENT WILL LICENSOR BE LIABLE TO YOU ON ANY LEGAL THEORY FOR ANY SPECIAL, INCIDENTAL, CONSEQUENTIAL, PUNITIVE OR EXEMPLARY DAMAGES ARISING OUT OF THIS LICENSE OR THE USE OF THE WORK, EVEN IF LICENSOR HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES.

7. Termination.

- a. This License and the rights granted hereunder will terminate automatically upon any breach by You of the terms of this License. Individuals or entities who have received Collective Works from You under this License, however, will not have their licenses terminated provided such individuals or entities remain in full compliance with those licenses. Sections 1, 2, 5, 6, 7, and 8 will survive any termination of this License.
- b. Subject to the above terms and conditions, the license granted here is perpetual (for the duration of the applicable copyright in the Work). Notwithstanding the above, Licensor reserves the right to release the Work under different license terms or to stop distributing the Work at any time; provided, however that any such election will not serve to withdraw this License (or any other license that has been, or is required to be, granted under the terms of this License), and this License will continue in full force and effect unless terminated as stated above.

8. Miscellaneous.

- a. Each time You distribute or publicly digitally perform the Work or a Collective Work, the Licensor offers to the recipient a license to the Work on the same terms and conditions as the license granted to You under this License.
- b. If any provision of this License is invalid or unenforceable under applicable law, it shall not affect the validity or enforceability of the remainder of the terms of this License, and without further action by the parties to this agreement, such provision shall be reformed to the minimum extent necessary to make such provision valid and enforceable.
- c. No term or provision of this License shall be deemed waived and no breach consented to unless such waiver or consent shall be in writing and signed by the party to be charged with such waiver or consent.

- d. This License constitutes the entire agreement between the parties with respect to the Work licensed here. There are no understandings, agreements or representations with respect to the Work not specified here. Licensor shall not be bound by any additional provisions that may appear in any communication from You. This License may not be modified without the mutual written agreement of the Licensor and You.

Bibliography

- ALEXANDER, H. R. (2003). Hyperthermia and its modern use in cancer treatment. *Cancer*, **98**, 219–221.
- ARPACI, V. S., SELAMET, A., AND KAO, S. H., *Introduction to Heat Transfer* (Prentice Hall, Upper Saddle River NJ, 2000).
- BEJAN, A., *Convection Heat Transfer* (John Wiley & Sons, New York NY, 1984).
- BERGMANN, G., GRAICHEN, F., ROHLMANN, A., VERDONSCHOT, N., AND VAN LENTHE, G. H. (2001). Frictional heating of total hip implants: Part 1: Measurements in patients. *Journal of Biomechanics*, **34**, 421–428.
- BOYCE, W. E. AND DIPRIMA, R. C., *Elementary Differential Equations and Boundary Value Problems* (John Wiley & Sons, New York NY, 1977), 3rd edition.
- BURMEISTER, L. C., *Convective Heat Transfer* (John Wiley & Sons, New York NY, 1983).
- CARRIER, G. F. AND PEARSON, C. E., *Partial Differential Equations: Theory and Technique* (Academic Press, New York NY, 1976).
- DUNKLE, R. V. (1954). Thermal radiation tables and applications. *Trans. ASME*, **76**, 549–549.
- GEBHART, B., JALURIA, Y., MAHAJAN, R. L., AND SAMMAKIA, B., *Buoyancy-Induced Flows and Transport* (Hemisphere, Washington DC, 1988).
- GILLMAN, L. AND MCDOWELL, R. H., *Calculus* (W. W. Norton and Co., New York NY, 1978), 2nd edition.
- INCROPERA, F. P. AND DEWITT, D. P., *Fundamentals of Heat and Mass Transfer* (John Wiley & Sons, New York NY, 2002), 5th edition.
- KAYS, W. M. AND CRAWFORD, M. E., *Convective Heat and Mass Transfer* (McGraw-Hill, New York NY, 1980), 2nd edition.

KREYSZIG, E., *Advanced Engineering Mathematics* (John Wiley & Sons, New York NY, 1988), 6th edition.

LANDER, E. S., LINTON, L. M., BIRREN, B., NUSBAUM, C., ZODY, M. C., BALDWIN, J., DEVON, K., DEWAR, K., DOYLE, M., FITZHUGH, W., FUNKE, R., GAGE, D., HARRIS, K., HEAFORD, A., HOWLAND, J., KANN, L., LEHOCZKY, J., LEVINE, R., MCEWAN, P., MCKERNAN, K., MELDRIM, J., MESIROV, J. P., MIRANDA, C., MORRIS, W., NAYLOR, J., RAYMOND, C., ROSETTI, M., SANTOS, R., SHERIDAN, A., SOUGNEZ, C., STANGE-THOMANN, N., STOJANOVIC, N., SUBRAMANIAN, A., WYMAN, D., ROGERS, J., SULSTON, J., AINSCOUGH, R., BECK, S., BENTLEY, D., BURTON, J., CLEE, C., CARTER, N., COULSON, A., DEADMAN, R., DELOUKAS, P., DUNHAM, A., DUNHAM, I., DURBIN, R., FRENCH, L., GRAFHAM, D., GREGORY, S., HUBBARD, T., HUMPHRAY, S., HUNT, A., JONES, M., LLOYD, C., MCMURRAY, A., MATTHEWS, L., MERCER, S., MILNE, S., MULLIKIN, J. C., MUNGALL, A., PLUMB, R., ROSS, M., SHOWNKEEN, R., SIMS, S., WATERSTON, R. H., WILSON, R. K., HILLIER, L. W., MCPHERSON, J. D., MARRA, M. A., MARDIS, E. R., FULTON, L. A., CHINWALLA, A. T., PEPIN, K. H., GISH, W. R., CHISSOE, S. L., WENDL, M. C., DELEHAUNTY, K. D., MINER, T. L., DELEHAUNTY, A., KRAMER, J. B., COOK, L. L., FULTON, R. S., JOHNSON, D. L., MINX, P. J., CLIFTON, S. W., HAWKINS, T., BRANSCOMB, E., PREDKI, P., RICHARDSON, P., WENNING, S., SLEZAK, T., DOGGETT, N., CHENG, J. F., OLSEN, A., LUCAS, S., ELKIN, C., UBERBACHER, E., FRAZIER, M., GIBBS, R. A., MUZNY, D. M., SCHERER, S. E., BOUCK, J. B., SODERGREN, E. J., WORLEY, K. C., RIVES, C. M., GORRELL, J. H., METZKER, M. L., NAYLOR, S. L., KUCHERLAPATI, R. S., NELSON, D. L., WEINSTOCK, G. M., SAKAKI, Y., FUJIYAMA, A., HATTORI, M., YADA, T., TOYODA, A., ITOH, T., KAWAGOE, C., WATANABE, H., TOTOKI, Y., TAYLOR, T., WEISSENBACH, J., HEILIG, R., SAURIN, W., ARTIGUENAVE, F., BROTTIER, P., BRULS, T., PELLETIER, E., ROBERT, C., WINCKER, P., ROSENTHAL, A., PLATZER, M., NYAKATURA, G., TAUDIEN, S., RUMP, A., YANG, H. M., YU, J., WANG, J., HUANG, G. Y., GU, J., HOOD, L., ROWEN, L., MADAN, A., QIN, S. Z., DAVIS, R. W., FEDERSPIEL, N. A., ABOLA, A. P., PROCTOR, M. J., MYERS, R. M., SCHMUTZ, J., DICKSON, M., GRIMWOOD, J., COX, D. R., OLSON, M. V., KAUL, R., RAYMOND, C., SHIMIZU, N., KAWASAKI, K., MINOSHIMA, S., EVANS, G. A., ATHANASIOU, M., SCHULTZ, R., ROE, B. A., CHEN, F., PAN, H. Q., RAMSER, J., LEHRACH, H., REINHARDT, R., MCCOMBIE, W. R., DE LA BASTIDE, M., DEDHIA, N., BLOCKER, H., HORNISCHER, K., NORDSIEK, G., AGARWALA, R., ARAVIND, L., BAILEY, J. A., BATEMAN, A., BATZOGLOU, S., BIRNEY, E., BORK, P., BROWN, D. G., BURGE, C. B., CERUTTI, L., CHEN, H. C., CHURCH, D., CLAMP, M., COPLEY, R. R., DOERKS, T., EDDY, S. R., EICHLER,

- E. E., FUREY, T. S., GALAGAN, J., GILBERT, J. G. R., HARMON, C., HAYASHIZAKI, Y., HAUSSLER, D., HERMJAKOB, H., HOKAMP, K., JANG, W. H., JOHNSON, L. S., JONES, T. A., KASIF, S., KASPRYZK, A., KENNEDY, S., KENT, W. J., KITTS, P., KOONIN, E. V., KORF, I., KULP, D., LANCET, D., LOWE, T. M., MCLYSAGHT, A., MIKKELSEN, T., MORAN, J. V., MULDER, N., POLLARA, V. J., PONTING, C. P., SCHULER, G., SCHULTZ, J. R., SLATER, G., SMIT, A. F. A., STUPKA, E., SZUSTAKOWKI, J., THIERRY-MIEG, D., THIERRY-MIEG, J., WAGNER, L., WALLIS, J., WHEELER, R., WILLIAMS, A., WOLF, Y. I., WOLFE, K. H., YANG, S. P., YEH, R. F., COLLINS, F., GUYER, M. S., PETERSON, J., FELSENFELD, A., WETTERSTRAND, K. A., PATRINOS, A., AND MORGAN, M. J. (2001). Initial sequencing and analysis of the human genome. *Nature*, **409**, 860–921.
- LEE, T. K., ZHONG, X. L., GONG, L., AND QUINN, R. (2003). Hypersonic aerodynamic heating prediction using weighted essentially nonoscillatory schemes. *Journal of Spacecraft and Rockets*, **40**, 294–298.
- MILLS, A. F., *Heat Transfer* (Prentice Hall, Upper Saddle River NJ, 1999), 2nd edition.
- ODEH, S. D., BEHNIA, M., AND MORRISON, G. L. (2000). Hydrodynamic analysis of direct steam generation solar collectors. *Journal of Solar Energy Engineering*, **122**, 14–22.
- ÖZİŞİK, M. N., *Heat Conduction* (John Wiley & Sons, New York NY, 1980).
- ÖZİŞİK, M. N., *Heat Transfer: A Basic Approach* (McGraw–Hill, New York NY, 1985).
- PANTON, R. L., *Incompressible Flow* (John Wiley & Sons, New York NY, 1984).
- PASTUKHOV, V. G., MAIDANIK, Y. F., VERSHININ, C. V., AND KORUKOV, M. A. (2003). Miniature loop heat pipes for electronics cooling. *Applied Thermal Engineering*, **23**, 1125–1135.
- POHLHAUSEN, K. (1921). Zur näherungsweise Integration der differential Gleichung der laminaren Reibungsschicht. *Zeitschrift für angewandte Mathematik und Mechanik*, **1**, 252–268.
- SCHLICHTING, H., *Boundary Layer Theory* (McGraw–Hill, New York NY, 1979), 7th edition.
- SIEGEL, R. AND HOWELL, J., *Thermal Radiation Heat Transfer* (Taylor and Francis, London, 2001), 4th edition.

- SINHA, S., KUMAR, S., AND KUMAR, N. (1998). Energy conservation in high-rise buildings: Changes in air conditioning load induced by vertical temperature and humidity profile in Delhi. *Energy Conversion and Management*, **39**, 437–440.
- TAYMAZ, I., ÇAKIR, K., GUR, M., AND MIMAROĞLU, A. (2003). Experimental investigation of heat losses in a ceramic coated diesel engine. *Surface and Coatings Technology*, **169**, 168–170.
- TEMA, *Standards of the Tubular Exchange Manufacturers Association* (TEMA, New York NY, 1978), 6th edition.
- VON KARMAN, T. (1921). Über laminare und turbulente Reibung. *Zeitschrift für angewandte Mathematik und Mechanik*, **1**, 233–252.
- WOOD, W. D., DEEM, H. W., AND LUCKS, C. F., *Thermal Radiative Properties* (Plenum Press, New York NY, 1964).

Index

- Biot number, 40, 44
- Bisection Method, 159
- boundary conditions, 13
 - adiabatic, 14
 - Dirichlet, 14, 149
 - Neumann, 14
 - no-slip, 55, 69, 71, 78, 85
 - Robbins, 14
- boundary layer approximation, 57, 91
- boundary layer equations, 59, 92
- Boussinesq approximation, 92
- buoyancy, 90

- Chain Rule of Calculus, 44, 49, 150
- characteristic length scale, 41
- Churchill–Chu correlation
 - horizontal cylinder, 98
 - vertical plate, 97
- circuit analogy, 17–23, 143
- conduction, 2
 - equation, 9–12
 - constant conductivity, 12
 - cylindrical coordinates, 21
 - one-dimensional, 13
 - steady, 13
 - one-dimensional, 16–17
 - circuit analogy, 17–23
- continuum assumption, 4–6
- control surface, 3
- control volume, 3
- convection, 2, 53–54
 - coefficient, 14, 18, 27, 53, 62
 - heat exchanger, overall, 103
 - infinite, 47
 - external, 66–67
 - internal, 77
 - natural, 90
- Couette flow, 85

- differential equation
 - separable, 38
- Dittus–Boelter correlation, 88
- DNA, 23

- Eckert number, 86
- eigenvalues, 45, 152
- emissive power, 120

- fins, 27–36
 - effectiveness, 34
 - efficiency, 35
 - general equation, 29
- Fourier number, 44, 149
- Fourier’s Law, 7, 17, 22, 28, 37, 50, 55, 81

- Gaussian Error Function, 50
- gel electrophoresis, 23
- Grashof number, 93

- heat exchanger, 99–100
 - ϵ -NTU analysis, 110–115
 - coefficient, 103
 - concentric tube, 99
 - correction factors, 109
 - fouling factor, 104
 - heat capacity rate, 111
 - LMTD analysis, 104–109
 - shell-and-tube, 99
 - thumbnail diagram, 100–102
- heat generation, 11, 23–27
- hydraulic diameter, 89
- hyperbolic trig functions, 30

- Ideal Gas Law, 6, 93
- initial conditions, 13
- irradiation, 121

- Kirchhoff’s Circuit Law, 145

- L’Hospitals rule, 108
- Laplacian operator, 13

- lumped capacitance analysis, 38, 149
- M-16 rifle, 2, 22
- Navier–Stokes Equations, 57, 85
- Newton’s Law of Cooling, 18, 27, 29, 33, 38, 53, 81, 105
- Newton’s Law of Viscosity, 55
- Nusselt number, 40, 63
- overall heat transfer coefficient, 103
- Peclet number, 74
- Prandtl number, 60, 86
- product solution, 50
- radiation, 2, 116–117
 - blackbody, 122
 - Planck’s Distribution, 123
 - Stefan–Boltzmann Law, 124
 - Wien’s Displacement Law, 124
 - diffuse, 121
 - emissivity, 126
 - gray body, 132
 - intensity, 119
 - Kirchhoff’s Law, 130
 - opaque body, 117, 128
 - re-radiating surface, 147
 - semi-transparent body, 117, 128
 - view factor, 134
 - reciprocity rule, 136
 - summation rule, 137
- radiosity, 122
- Rayleigh number, 96
- resistors
 - in parallel, 19
 - in series, 19
- Reynolds number, 56, 60, 78
- Reynolds–Colburn analogy, 63
- Sieder–Tate correlation, 88
- similarity transform, 48, 67, 94
- skin friction factor, 62, 64, 68, 73
- solid angle, 117
- stream function, 67, 95
- Taylor series, 9, 28
- temperature, 1
 - absolute and relative, 6
 - film, 66
 - gradient, 7–8
- thermal
 - conductivity, 7, 8
 - contact resistance, 19
 - diffusivity, 8, 37
 - resistance, 17, 103
 - thermal expansion coefficient, 92
 - thumbnail diagram, 100–102
 - volumetric heat capacity, 8