JME 3710
Midterm Quiz: November 3, 2011 **Name:** ________________________

Solutions

1. (10 pts) To estimate the thermal conductivity, k, of a material, a tubular test device is constructed of the material and a heating element is placed in its center (shaded area, Fig. 1). The element provides $100\,W$ of thermal energy per meter length of the tube. The tube is long, so that we assume heat transfer only in the radial direction. Inner and outer radii are $r_i = 0.26\,m$ and $r_o = 0.34\,m$, respectively. Thermocouples indicate the temperatures at the inner and outer surfaces of the tube are $T_i = 339\,K$ and $T_o = 311\,K$, respectively. Calculate k in units of $W/(mK)$. *Hint:* There is no energy generation within the material itself.

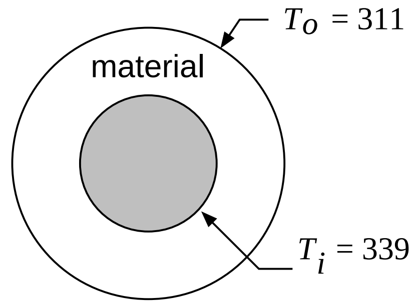

Fig. 1: Cylinder with heat element.

Solution: Because there is no energy generation in the material itself, we can model conduction using the circuit analogy. We have direct temperature measurements at both surfaces, so this configuration reduces to a single resistor representing the material with the 2 end nodes representing the surface temperatures. Consequently, we write the simple circuit–conduction analogy in units of power per unit length of tubing as

$$\frac{q}{L} = 100\,\frac{W}{m} = \frac{2\,\pi\,k\,\Delta T}{\ln(r_o/r_i)}\,,$$

where $r_o = 0.34\,m$ and $r_i = 0.26\,m$ and $\Delta T = 339 - 311 = 28\,K$. Solving for k, we find

$$k = \frac{q}{L} \times \frac{\ln(r_o/r_i)}{2\,\pi\,\Delta T} = 100\,\frac{W}{m} \times \frac{\ln(0.34/0.26)}{2 \cdot \pi \cdot 28}\,K^{-1} = 0.153\,\frac{W}{m\,K}\,.$$

2. Forced convection occurs in a duct having the cross–section of a right triangle where the three side lengths are L, L, and $\sqrt{2}\,L$ (Fig. 2), where $L = 0.1\,m$. The fluid has a viscosity of $\nu = 10^{-5}\,m^2/s$, a conductivity of $k = 0.02\,W/(m\,K)$, and is flowing at an average velocity of $\overline{u} = 0.2\,m/s$. Assume the flow is laminar and fully–developed. In this particular case, the Nusselt number can be taken as $Nu = 3.5$.

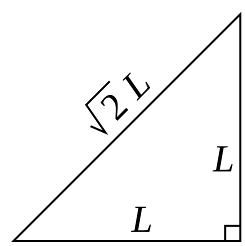

Fig. 2: *Right–triangle duct.*

(a) (10 pts) Determine the value of the hydraulic diameter, D_H, which is the length scale for this configuration.

Solution: We calculate this directly from the definition of hydraulic diameter as

$$D_h = \frac{4\,A}{P} = \frac{4}{L\left(1+1+\sqrt{2}\,\right)}\,\frac{L^2}{2} = \frac{2\,L}{2+\sqrt{2}} \approx 0.5858\,L\,,$$

so that, by substituting $L = 0.1\,m$, we find $D_h = 0.05858\,m$.

(b) (10 pts) Demonstrate that the flow is indeed laminar using the proper dimensionless argument.

Solution: The Reynolds number is the relevant dimensionless parameter and its value is

$$Re \;=\; \frac{\overline{u}\, D_h}{\nu} \;=\; \frac{0.2 \cdot 0.05858}{10^{-5}} \;\approx\; 1172\,,$$

which is well below the critical value of around 2100 to 2300.

(c) (10 pts) Calculate the convection heat transfer coefficient, h.

Solution: Mindful that D_H is the length scale for this problem, the Nusselt number is defined as $Nu = hD_H/k$. Therefore,

$$h \;=\; \frac{Nu \cdot k}{D_h} \;=\; \frac{3.5 \cdot 0.02}{0.05858} \;=\; 1.19495 \;\approx\; 1.2\, \frac{W}{m^2\, K}\,.$$

3. The ACME Window Corporation (AWC) has just received a sample of a newly synthesized, unusual fluid that the vendor claims will give remarkable results when used as the "filler" in sealed double–pane windows. They provide you data showing that the density ρ relies on temperature T according to

$$\rho \;=\; \rho_o\, e^{-\,\nu\,\alpha\, T/C}\,,$$

where ρ_o is a fixed reference density, ν and α are the constant kinematic viscosity and constant thermal diffusivity, respectively, and C is a constant having a value of $C = 0.001\, m^4\, K/sec^2$. Your boss assigns you to evaluate the vendor's claims from the standpoint of how well the fluid will limit natural convection heat transfer between the two panes.

(a) (10 pts) As a competent heat transfer engineer, you realize that you should first determine the volumetric thermal expansion coefficient of this new fluid, β. Do that here. (Your solution should contain ν and α.)

Solution: Since temperature is the only variable that affects ρ, we can evaluate β directly from its definition

$$\beta \;=\; -\,\frac{1}{\rho}\,\left(\frac{\partial \rho}{\partial T}\right)\Bigg|_P \;=\; -\,\frac{1}{\rho_o\, e^{-\,\nu\,\alpha\,T/C}}\,\left(\rho_o\, e^{-\,\nu\,\alpha\,T/C}\,\left[-\,\frac{\nu\,\alpha}{C}\right]\right) \;=\; \frac{\nu\,\alpha}{C} \;=\; 1000\,\nu\,\alpha$$

in units of K^{-1}.

(b) (10 pts) You recall from your heat transfer class that natural convection in such enclosures will be essentially negligible if the Rayleigh number is small, i.e. $Ra_L < 10^3$. What conclusion will you draw in your report regarding the importance of natural convection if your windows are designed according to a length scale (gap size) of $L = 0.01\, m$ and for maximum temperature differences of $\Delta T \sim 50\, ^oC$?

Solution: The Rayleigh number Ra_L is the product of the Grashof and Prandtl numbers

$$Ra_L \;=\; Gr_L\, Pr \;=\; \frac{g\,\beta\,\Delta T\, L^3}{\nu^2}\,\frac{\nu}{\alpha} \;=\; \frac{g\,\beta\,\Delta T\, L^3}{\nu\,\alpha}.$$

Using β for this specific fluid, we find

$$Ra_L = \frac{g \cdot 1000 \cdot \nu\,\alpha\,\Delta T\,L^3}{\nu\,\alpha} = 1000 \cdot g\,\Delta T\,L^3\,,$$

for which we can estimate the size of Ra_L as

$$Ra_L \approx 1000 \cdot 9.8 \cdot 50 \cdot 0.01^3 \approx 0.5\,.$$

Therefore, since $Ra_L \ll 10^3$ you will report that this fluid effectively eliminates natural convection. Only conduction would be important.

4. Heat conduction is one–dimensional in the x direction in a particular slab of unit thickness (Fig. 3). Its conductivity is k and the boundary temperatures at $x = 0$ and $x = 1$ are both T_b. The material is unusual in that it has a variable internal heat generation of $\dot{q}(x) = (1 - x)\,\dot{q}_0$, where $\dot{q}_0$ is a prescribed constant. The conduction equation governing the temperature distribution $T(x)$ for this problem is

$$\frac{d^2T}{dx^2} = -\frac{\dot{q}(x)}{k}\,.$$

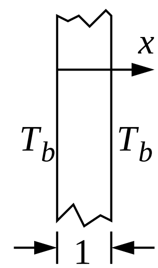

Fig. 3: Slab.

(a) (10 pts) Briefly explain why the circuit analogy of conduction analysis cannot be used here.

Solution: In the Cartesian coordinate system, the circuit analogy depends upon a linear variation in temperature, or equivalently, a constant temperature gradient, so as to have a form that is consistent with the circuit analogy, where $q \propto \Delta T$. Problems having heat generation violate this restriction.

(b) (10 pts) Integrate the governing equation to obtain the general solution for $T(x)$. Use C_1 and C_2 for your integration constants. *Hint:* Note that the heat generation is *not* constant within the domain.

Solution: Substituting the given form of the heat generation and integrating twice, we find

$$\begin{aligned}
\frac{d^2T}{dx^2} &= -\frac{(1-x)\,\dot{q}_0}{k} = \frac{(x-1)\,\dot{q}_0}{k} \\
\frac{dT}{dx} &= \left(\frac{x^2}{2} - x\right)\frac{\dot{q}_0}{k} + C_1 \\
T(x) &= \left(\frac{x^3}{6} - \frac{x^2}{2}\right)\frac{\dot{q}_0}{k} + C_1 x + C_2\,.
\end{aligned}$$

(c) (10 pts) Given the boundary temperatures of T_b at $x = 0$ and $x = 1$, solve for C_1 and C_2 to obtain the exact solution for $T(x)$ for this problem. *Hint:* Your solution should have the form $T(x) = (x^3/a_3 + x^2/a_2 + x/a_1)\dot{q}_0/k + T_b$, where a_1, a_2, and a_3 are integers that you determine.

Solution: For $x = 0$, we find

$$T(0) \;=\; T_b \;=\; \left(\frac{0^3}{6} \;-\; \frac{0^2}{2}\right)\,\frac{\dot{q}_0}{k} \;+\; C_1 \cdot 0 \;+\; C_2\,, \quad \text{implying} \quad C_2 \;=\; T_b\,,$$

and for $x = 1$, and using (substituting) $C_2 = T_b$, we then find

$$T(1) \;=\; T_b \;=\; \left(\frac{1^3}{6} \;-\; \frac{1^2}{2}\right)\,\frac{\dot{q}_0}{k} \;+\; C_1 \cdot 1 \;+\; T_b\,, \quad \text{implying} \quad C_1 \;=\; \frac{\dot{q}_0}{3\,k}\,.$$

Therefore, the exact solution can be written as

$$T(x) \;=\; \left(\frac{x^3}{6} \;-\; \frac{x^2}{2} \;+\; \frac{x}{3}\right)\,\frac{\dot{q}_0}{k} \;+\; T_b\,.$$

(d) (10 pts) Demonstrate that there is a maximum temperature, T_{max}, that occurs within $0 \le x \le 1$ and determine its location, x_{max}. (You do not have to determine T_{max} itself.)

Solution: To identify any extrema, we take the first derivative of $T(x)$ and set it to zero, finding

$$\frac{dT}{dx} \;=\; \frac{x^2}{2} \;-\; x \;+\; \frac{1}{3} \;=\; 0\,.$$

This expression does not appear to be readily factorable, so we use the quadratic formula to obtain

$$x \;=\; \frac{1 \pm \sqrt{1 - 4 \cdot (1/2) \cdot (1/3)}}{2 \cdot (1/2)} \;=\; 1 \;\pm\; \sqrt{1 - \frac{2}{3}} \;=\; 1 \;\pm\; \sqrt{\frac{1}{3}}\;.$$

One of these values is clearly outside the problem domain of $0 \le x \le 1$, so it appears the other *may* be our solution. However, we have to confirm that this extreme is actually a maximum. We do that using the standard second derivative test:

$$\left.\frac{d^2T}{dx^2}\right|_{x=1-\sqrt{1/3}} \;=\; \left.(x \;-\; 1)\right|_{x=1-\sqrt{1/3}} \;=\; \text{a negative value.}$$

Therefore, this value of x is indeed a maximum, so that

$$x_{max} \;=\; 1 \;-\; \sqrt{\frac{1}{3}} \;=\; 1 \;-\; \frac{1}{\sqrt{3}} \;=\; 1 \;-\; \frac{\sqrt{3}}{3} \;\approx\; 0.423\,.$$

Any of these versions of the answer is acceptable.