**JME 3700**
**Final Quiz: May 10, 2011** **Name:** ____________________

**Solutions**

1. (10 pts) One of the important areas that arises in civil engineering is "building aerodynamics", e.g. pressure distribution over a high–rise building and the structural response due to flow separation, oscillatory forcing, etc. Dimensional analysis for a particular high–rise of height $h$ indicates that the local gage pressure $P$ on the surface of the building has the *functional form*

$$\frac{P}{\rho\, V^2} = \Phi\left(\frac{h}{w}\,,\,\frac{h}{l}\right),$$

where $\rho$ is air density, $V$ is wind speed, and $w$ and $l$ are the width and length of the building, respectively. In other words, this equation can be read as "the dimensionless pressure is a function of two dimensionless geometric parameters". If you are testing a *scale model* of this building in a wind tunnel, determine the relationship between the full–scale local gage pressure $P$ and the corresponding pressure on the wind tunnel model $P_m$. Use the subscript $m$ for variable(s) that refer to the wind tunnel model and assume air density in the wind tunnel is identical to air density for flow around the actual full–scale building.

*Solution:* If the model is "to scale" then all the dimensionless geometric parameters for the full–scale building are the same as the corresponding ones for the model. The only condition for complete similarity is then

$$\frac{P}{\rho\, V^2} = \frac{P_m}{\rho_m\, V_m^2}\,,$$

for which we find

$$P = \left(\frac{V}{V_m}\right)^2 P_m\,,$$

since the air density is the same for both cases.

2. (5 pts) *Briefly* explain the physical interpretation & significance of the Reynolds number and the implications for laminar versus turbulent flows. You may give an example in your explanation.

*Solution:* The Reynolds number essentially quantifies the importance of inertial effects versus viscous effects, i.e.

$$Re = \frac{\text{inertial effects}}{\text{viscous effects}}\,,$$

so that low $Re$ implies viscous effects dominate and any disturbances will tend to be damped–out. Those flows will tend to remain laminar. Inertial effects dominate high $Re$ flows so that disturbances tend *not* to be damped, leading usually to turbulent flow. For example, pipe flows are generally turbulent for $Re > 2100$, otherwise they're laminar. The corresponding threshold for flat plate flow is about 500,000.

3. Gravity–driven flow occurs in a rectangular–shaped duct that is extremely wide compared to its height, i.e. $y < a \ll b$ (Fig. 1). The Manning equation is a good description of this flow and the duct is made of corrugated metal having a Manning factor of $n = 0.022\, s/m^{1/3}$ and has a slope of $S_0 = 0.0014$.

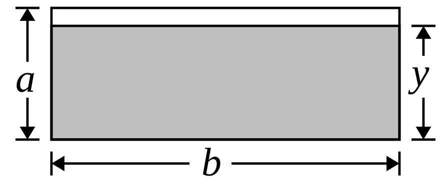


Fig. 1: *Rectangle channel*

(a) (10 pts) If the depth is $y = 2\,m$ and the gravitational acceleration is $g = 9.8\,m/s^2$, determine whether the flow is sub–critical or super–critical using the appropriate dimensionless parameter. *Hint:* Make use of the large aspect ratio to eliminate the unknown width $b$ in calculating $R_h$.

*Solution:* The area and wetted perimeter are $A = yb$ and $P = 2y+b$, respectively, whereby the hydraulic radius, $R_h = A/P$, is

$$R_h = \frac{y\,b}{2\,y+b} \approx y = 2\,m\,,$$

since $2y \ll b$. The Manning formula is then

$$\bar{u} = \frac{R_h^{2/3}\,S_0^{1/2}}{n} = \frac{2^{2/3}\,\sqrt{0.0014}}{0.022} \approx 2.7\,m/s\,.$$

Finally, the Froude number is

$$Fr = \frac{\bar{u}}{\sqrt{g\,y}} = \frac{2.7}{\sqrt{9.8\cdot 2}} \approx 0.61\,,$$

whereby we see the flow is sub–critical, since $Fr < 1$.

(b) (10 pts) Calculate the volumetric flow rate, $Q$, in terms of the unknown width $b$, i.e. your answer should have the form of some number times $b$.

*Solution:* The volumetric flow rate is also calculated from Manning theory. Since we already have the average velocity, we can just multiply by the flow area

$$Q = \bar{u}\,A = \bar{u}\,y\,b = 2.7\cdot 2\,b = 5.4\,b\,.$$

(c) (10 pts) In the previous parts, the flow filled 95% of the cross–section, i.e. $y = 0.95\,a$. Now suppose the flow depth is increased so that the flow completely fills the cross–section (the flow is still gravity–driven), while all other parameters remain the unchanged. Calculate the new flow rate and explain why it somewhat counter–intuitively is less than for the previous part. Again, make use of the large aspect ratio to calculate $R_h$.

*Solution:* We see that $a = y/0.95 = 2/0.95 \approx 2.105\,m$, so that

$$R_h = \frac{A}{P} = \frac{a\,b}{2\,a+2\,b} = \frac{a\,b}{2\,(a+b)} \approx \frac{a}{2} = \frac{2.105}{2} \approx 1.053\,.$$

since $a \ll b$. The Manning formula is then

$$Q = \frac{A\, R_h^{2/3}\, S_0^{1/2}}{n} = \frac{a\, b\, R_h^{2/3}\, S_0^{1/2}}{n} = \frac{2.105 \cdot b \cdot 1.053^{2/3}\, \sqrt{0.0014}}{0.022} \approx 3.7\, b\,.$$

This is clearly less than the figure above having the coefficient 5.4. Although the flow area is *slightly* higher for this case, which increases volumetric flow rate, the hydraulic radius is substantially less because the flow is now in full contact with the "roof" of the duct. Physically, the added shear stress from this contact with a no–slip wall is a much bigger factor in reducing the flow rate.

4. A simple 2–dimensional potential flow called the *doublet* has a velocity potential function given by $\phi = C\, \cos\theta/r$, where $C$ is a constant and $\theta$ and $r$ are circular coordinates (Fig. 2). Helpful operators in the cylinderical coordinate system are

$$\nabla\phi = \frac{\partial \phi}{\partial r}\,\hat{r} + \frac{1}{r}\,\frac{\partial \phi}{\partial \theta}\,\hat{\theta}$$

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\,\frac{\partial (r\, u_r)}{\partial r} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta}$$

and helpful differentials are

$$\frac{d}{d\theta}\Big(\sin\theta\Big) = \cos\theta \quad \text{and} \quad \frac{d}{d\theta}\Big(\cos\theta\Big) = -\,\sin\theta\,.$$

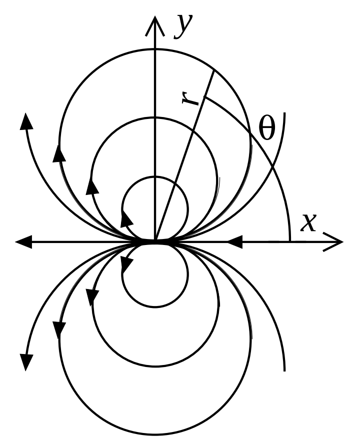


Fig. 2: *The doublet.*

(a) (10 pts) Aside from the condition of incompressibility, identify the two main restrictions required for the potential flow model to apply.

*Solution:*

1. inviscid flow
2. irrotational flow

(b) (10 pts) Derive $u_r$ and $u_\theta$, the velocity components in the $r$ and $\theta$ directions, respectively, using the potential function.

*Solution:*

$$u_r = \frac{\partial \phi}{\partial r} = -\,C\,\frac{\cos\theta}{r^2} \qquad \text{and} \qquad u_\theta = \frac{1}{r}\,\frac{\partial \phi}{\partial \theta} = -\,C\,\frac{\sin\theta}{r^2}$$

(c) (10 pts) Show that the flow satisfies the differential form of conservation of mass.

*Solution:* In this case, conservation of mass is expressed as

$$\nabla \cdot \mathbf{V} = \frac{1}{r}\,\frac{\partial (r\, u_r)}{\partial r} + \frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} = 0\,.$$

Using the velocity components from the previous section, we find

$$\frac{1}{r}\,\frac{\partial (r\, u_r)}{\partial r} = \frac{1}{r}\,\frac{\partial}{\partial r}\left(-\,C\,\frac{\cos\theta}{r}\right) = \frac{1}{r}\,C\,\frac{\cos\theta}{r^2} = C\,\frac{\cos\theta}{r^3}$$

$$\frac{1}{r}\,\frac{\partial u_\theta}{\partial \theta} \;=\; \frac{1}{r}\,\frac{\partial}{\partial \theta}\left(-\,C\,\frac{\sin\theta}{r^2}\right) \;=\; -\,C\,\frac{\cos\theta}{r^3}\,,$$

which clearly sum to zero, thus satisfying conservation of mass.

5. A tube having a rectangular cross–section of $s$ by $2s$ (where $s = 0.02\ m$) is fashioned into a siphon to transfer water from a very large reservoir at a design volumetric flow rate of $Q = 5.7 \times 10^{-5}\ m^3/s$ (Fig. 3). The crest height of the tube above the reservoir surface is $H$ and the flow is taken to be fully–developed throughout. Gravitational acceleration is $g = 9.8\ m/s^2$ in the negative $y$–direction. The viscous (head) loss from the inlet to the crest is modelled as

$$h_L \;=\; K\,\frac{\bar{u}^2}{2\,g}\,,$$

where $K > 0$ is an empirical constant and $\bar{u}$ is the average velocity.

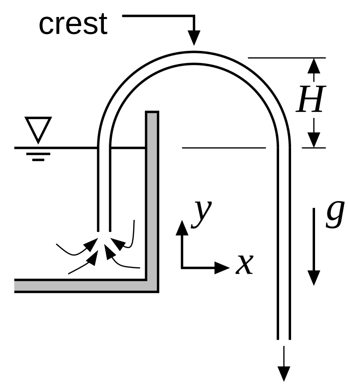


Fig. 3: *Water siphon*

(a) (10 pts) Given material properties of water: $\nu = 1 \times 10^{-6}\ m^2/s$ (kinematic viscosity) and $\rho = 10^3\ kg/m^3$ (density), determine whether the flow is laminar or turbulent using the appropriate dimensionless argument.

*Solution:* Here, we use the Reynolds number based on the length scale of the *hydraulic diameter* because the cross–section is not circular. The average velocity at the design flow rate is

$$\bar{u} \;=\; \frac{Q}{s \cdot 2\,s} \;=\; \frac{Q}{2\,s^2} \;=\; \frac{5.7 \times 10^{-5}}{0.02 \cdot 0.04} \;=\; 0.07125\ m/s$$

and the hydraulic diameter is

$$D_h \;=\; \frac{4\,A}{P} \;=\; \frac{4 \cdot s \cdot 2\,s}{s\,+\,2\,s\,+\,s\,+\,2\,s} \;=\; \frac{4\,s}{3} \;\approx\; 0.02666667\ m\,,$$

whereby the Reynolds number is

$$Re \;=\; \frac{\bar{u}\,D_h}{\nu} \;=\; \frac{Q\,4\,s}{2\,s^2 \cdot 3\,\nu} \;=\; \frac{2\,Q}{3\,\nu\,s} \;=\; \frac{2 \cdot 5.7 \times 10^{-5}}{3 \cdot 1 \times 10^{-6} \cdot 0.02} \;=\; 1900\,.$$

Since this is somewhat below the critical value of around 2100, the flow is evidently laminar.

(b) (10 pts) One of the design aspects of a siphon is the "jump–over" height, i.e. the vertical elevation, $H$, that the crest of the siphon must clear. This is limited by the phenomenon of *cavitation*. That is, if the static pressure is low enough at the crest, the liquid will start to gassify, thereby "breaking" the siphon. Use a generalized Bernoulli argument to derive an expression for $H$ (no numbers) assuming the flow breaks when the static pressure in the crest drops to 0. *Hint:* Work the problem in terms of absolute pressure, taking atmospheric pressure at the reservoir surface as $P_{atm}$.

*Solution:* The generalized Bernoulli equation can be written between the reservoir surface and the siphon's crest as

$$\frac{P_R}{\rho\, g} + \frac{V_R^2}{2\, g} + z_R - h_L = \frac{P_C}{\rho\, g} + \frac{V_C^2}{2\, g} + z_C \, .$$

Let $P_R = P_{atm}$, as stated in the hint, $V_R = 0$ because the reservoir is large, and $z_R = 0$ as the vertical datum. Also, $P_C = 0$ is the stated cavitation threshold, $V_C = \bar{u}$ is the average velocity (constant) throughout the siphon, and $z_C = H$ is the actual crest height. Then, using the loss model above, we find

$$\frac{P_{atm}}{\rho\, g} - K\, \frac{\bar{u}^2}{2\, g} = \frac{\bar{u}^2}{2\, g} + H \, .$$

Solving for $H$, we find

$$H = \frac{P_{atm}}{\rho\, g} - (K+1)\, \frac{\bar{u}^2}{2\, g} \, .$$

(c) (5 pts) The "jump–over" height reaches a theoretical maximum, $H_{max}$, as the siphon velocity becomes very small. Assuming an atmospheric pressure of $P_{atm} = 1.01 \times 10^5\, N/m^2$ and using the previous result, show that $H_{max} \approx 10.3\, m$ for water.

*Solution:* As $\bar{u}$ becomes small, the second term in the expression for $H$ becomes negligible because of the square of $\bar{u}$. Consequently,

$$H_{max} \rightarrow \frac{P_{atm}}{\rho\, g} = \frac{101,000}{1000 \cdot 9.8} \approx 10.3\, m \, .$$